Ordinary Differential Equations 2
Pre-Test Homework

Due: 4 pm on Tuesday 12 November 2019

Solution> | <Solution

1. (Based on Q1 from 2016-17 exam)

   Consider the following autonomous vector field on $\mathbb{R}^2$:

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy$$

   Give conditions on a, b, c, d for which:

   (a) there are no periodic orbits.

   (b) all orbits are periodic.

   (c) both the x- and y-axes (i.e., the lines $x = 0$ and $y = 0$) are invariant manifolds.

   You do *not* have to give all possible conditions on the constants in order for the dynamical condition to be satisfied. One condition will be sufficient, but you must justify your answer.

   Solution>

   (a) A linear ODE in $\mathbb{R}^2$ can have a (non-equilibrium) periodic orbit if and only if the origin is a centre, which in turn requires that the eigenvalues be purely imaginary, which in turn requires that the trace of the matrix be zero. Thus $a + d \neq 0$ is sufficient to eliminate periodic orbits.

   (b) Here we want the origin to be a centre, in other words the eigenvalues to be purely imaginary. An example is $a = d = 0$ and $b = -c$.

   (c) We need $(1, 0)$ and $(0, 1)$ to be eigenvectors so we need $b = c = 0$.

   <Solution

2. (Based on Q3 from 2017-18 exam)

   Consider the following autonomous vector field on $\mathbb{R}^2$ with $\omega > 0$ being a parameter:

$$\dot{x} = -\omega y - \frac{x}{2}\left(x^2 + y^2 - \frac{2}{3}(x^2 + y^2)^{\frac{3}{2}}\right),$$
$$\dot{y} = \omega x - \frac{y}{2}\left(x^2 + y^2 - \frac{2}{3}(x^2 + y^2)^{\frac{3}{2}}\right).$$

   (a) Show that $(x, y) = (0, 0)$ is an equilibrium.

(b) Transform the vector field to polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$.

Hint: First show that

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y}, \\ r^2\dot{\theta} &= x\dot{y} - y\dot{x} \end{aligned}$$

and use that to write the original ODE in terms of $r, \theta, \dot{r}, \dot{\theta}$.

Solution>

(a) $(x, y) = (0, 0)$ implies $(\dot{x}, \dot{y}) = (0, 0)$. Thus $(0, 0)$ is an equilibrium.

(b) Note that $x^2 + y^2 = r^2$. We calculate

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} = x\left(-\omega y - \frac{x}{2}\left(r^2 - \frac{2}{3}r^3\right)\right) + y\left(\omega x - \frac{y}{2}\left(r^2 - \frac{2}{3}r^3\right)\right) = -r^4\left(\frac{1}{2} - \frac{1}{3}r\right), \\ r^2\dot{\theta} &= x\dot{y} - y\dot{x} = x\left(\omega x - \frac{y}{2}\left(r^2 - \frac{2}{3}r^3\right)\right) - y\left(-\omega y - \frac{x}{2}\left(r^2 - \frac{2}{3}r^3\right)\right) = \omega r^2. \end{aligned}$$

Thus, in polar coordinates we have,

$$\begin{aligned} \dot{r} &= -r^3\left(\frac{1}{2} - \frac{1}{3}r\right), \\ \dot{\theta} &= \omega. \end{aligned}$$

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