UNIVERSITY OF BRISTOL

School of Mathematics

Ordinary Differential Equations 2

Mock Exam 2

Solution>	<Solution

17 December 2018 45 minutes

This paper contains **four** questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. (**8 marks**)

   Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = y - x\ f(x, y),$$
$$\dot{y} = -x - y\ f(x, y),$$

   where $f\colon \mathbb{R}^2 \to \mathbb{R}$ has a Taylor expansion around $(0, 0)$ and satisfies $f(0, 0) = 0$.

   (a) (**4 marks**) Show that $(0, 0)$ is stable if f is non-negative in a neighbourhood around the origin.

   (b) (**4 marks**) Under what conditions on f is $(0, 0)$ asymptotically stable?

   Solution>

   First we notice that $(0, 0)$ is an equilibrium of the ODE. Let $V(x, y) = \frac{1}{2}(x^2 + y^2)$; notice that V is non-negative and vanishes only at $(0, 0)$. We calculate

$$\dot{V} = x\dot{x} + y\dot{y} = x(y - xf) + y(-x - yf) = -(x^2 + y^2)f.$$

   (a) Since $\dot{V} \leqslant 0$ when f is non-negative, it follows from Lyapunov's Theorem that $(0, 0)$ is stable.

   (b) Following the reasoning above, for asymptotic stability we need $f > 0$ in a neighbourhood of the origin $(0, 0)$ (excluding $(0, 0)$).

   <Solution

2. (**10 marks**)

   Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = -y - x^3,$$
$$\dot{y} = x^5.$$

   Use the function $V(x, y) = x^6 + 3y^2$ to show that all solutions of the ODE approach $(0, 0)$ as $t \to \infty$.

   Solution>

   Let $c > 0$ and consider the closed bounded set $\mathcal{M} := \{(x, y)\ |V(x, y) \leqslant c\}$ (**2 marks**). Since

$$\dot{V} = 6x^5\dot{x} + 6y\dot{y} = 6x^5(-y - x^3) + 6yx^5 = -6x^8 \leqslant 0,$$

   $\mathcal{M}$ is positively invariant (**2 marks**).

   $\dot{V}$ vanishes precisely on the y-axis (**2 marks**). From the ODE the only point on the y-axis that remains on the y-axis is $(0, 0)$ (**2 marks**). Thus by the LeSalle Invariance Principle all trajectories in $\mathcal{M}$ approach $(0, 0)$ as $t \to \infty$. Since c can be chosen to be arbitrarily large this conclusion holds for all trajectories. (**2 marks**)

   <Solution

3. (**4 marks**)

How many periodic orbits exist for the ODE

$$\dot{x} = xe^{-x},$$
$$\dot{y} = 1 + x + y^2,$$

where $(x, y) \in \mathbb{R}^2$? You must justify your answer.

Solution>

First we determine the equilibrium points: $\dot{x} = 0 \implies x = 0$. With this condition, $\dot{y} = 1 + y^2 > 0$. Thus there are no equilibrium points. It follows, from the Index Theorem, that there are no periodic orbits.

<Solution

4. (**8 marks**)

Consider the following ODE for $x \in \mathbb{R}$:

$$\dot{x} = \mu x - \sin x,$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) (**3 marks**)

Classify the bifurcation at $\mu = 1$.

Solution>

Since the slope of sin at the origin is 1, when $\mu \geqslant 1$ the only equilibrium is at $x = 0$, which is unstable. When $\mu = 1 + \epsilon$ two new unstable equilibria appear at $x = \pm\delta$. Thus there is a sub-critical pitchfork at $\mu = 1$.

<Solution

(b) (**5 marks**)

Identify the largest $\mu < 1$ for which the system has a bifurcation. What kind of a bifurcation is it?

Solution>

From a figure it is clear that the largest $\mu < 1$ for which a bifurcation occurs is at $\mu \approx \frac{2}{5\pi}$ which satisfies both $\mu x_0 = \sin x_0$ and $\mu = \cos x_0$ (**3 marks**). It is easy to see that this is a saddle-node bifurcation (**2 marks**).

<Solution

End of examination.