UNIVERSITY OF BRISTOL

School of Mathematics

Ordinary Differential Equations 2
Mock Exam

9 November 2018 45 minutes

This paper contains **three** questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. (**15 marks**)

Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= \lambda x, \\ \dot{y} &= x + \lambda y, \end{aligned}$$

where $\lambda > 0$.

Solution> First we note that the system can be rewritten as

$$\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix}, \tag{1}$$

which is in canonical form. <Solution

(a) (**4 marks**)

Determine the equilibrium point(s) and its/their stability.

Solution> From (1), the equilibrium is $(0, 0)$ and, since $\lambda > 0$ it is unstable. <Solution

(b) (**6 marks**)

Compute the flow generated by the ODE.

Solution> From (1),

$$\begin{pmatrix} y(x) \\ x(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ x_0 \end{pmatrix}.$$

Thus the flow is,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \phi_t(x_0, y_0) = e^{\lambda t} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

<Solution

(c) (**5 marks**)

Draw the phase portrait.

2. (**5 marks**)

Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= x \ f(x, y), \\ \dot{y} &= y \ g(x, y), \end{aligned}$$

where $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ are smooth functions.

Show that each of the four quadrants, i.e. each of the sets

$$\begin{aligned} &\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}, \\ &\{(x, y) \in \mathbb{R}^2 \mid x > 0, y < 0\}, \\ &\{(x, y) \in \mathbb{R}^2 \mid x < 0, y > 0\}, \\ &\{(x, y) \in \mathbb{R}^2 \mid x < 0, y < 0\}, \end{aligned}$$

is an invariant set.

Solution> Since $\dot{x} = 0$ when $x = 0$, no trajectory can cross the y-axis. Similarly, since $\dot{y} = 0$ when $y = 0$, no trajectory can cross the x-axis. It follows that each quadrant is an invariant set. <Solution

3. (**10 marks**)

Which of the following trajectories on $\mathbb{R}^2$ can be solutions of a first-order autonomous linear ODE?

(a) $(x(t), y(t)) = (3e^t + e^{-t}, e^{2t})$.
(b) $(x(t), y(t)) = (3e^t + e^{-t}, e^t)$.
(c) $(x(t), y(t)) = (3e^t + e^{-t}, te^t)$.
(d) $(x(t), y(t)) = (3e^t, t^2e^t)$.
(e) $(x(t), y(t)) = (e^t + 2e^{-t}, e^t + 2e^{-t})$.

You must justify your answer.

Solution> From the canonical forms we see that the solution of a first-order two-dimensional autonomous linear ODE is one of the following:

- A linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.
- A linear combination of $e^{\lambda t}$ and $te^{\lambda t}$.
- A linear combination of $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$.

This eliminates (3a), (3c) and (3d).

For (3e), we see that $x = y$ so the solution lies in a one-dimensional subspace of $\mathbb{R}^2$. The solution of a first-order one-dimensional autonomous linear ODE must be of the form $e^{\lambda t}$, so (3e) is eliminated as well.

We conclude that only (3b) is the solution of a first-order two-dimensional autonomous linear ODE. <Solution

End of examination.