Chapter 4

1. $(T^{-1}\Lambda T)^k = T^{-1}\Lambda^k T$

The relation is true for $k=1$

$$T^{-1}\Lambda T = T^{-1}\Lambda T$$

Assume that it is true for $k=n$

$$(T^{-1}\Lambda T)^n = T^{-1}\Lambda^n T$$

and show that this implies that it is true for $k=n+1$

$$(T^{-1}\Lambda T)^{n+1} = (T^{-1}\Lambda T)^n (T^{-1}\Lambda T)$$

By the induction hypothesis $\rightarrow$ $= (T^{-1}\Lambda^n T)(T^{-1}\Lambda T)$

$$= T^{-1}\Lambda^{n+1} T \quad \checkmark$$

2. $\frac{d}{dt} e^{At} = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h}$

$$e^{A(t+h)} - e^{At} = e^{At}\left(e^{Ah} - \mathbb{1}\right)$$

Using the exponential Series

$$e^{Ah} - \mathbb{1} = Ah + \frac{1}{2!} A^2 h^2 + \cdots$$

Therefore

$$\lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h}$$

$$= \lim_{h \to 0} e^{At} \frac{1}{h}\left(Ah + \frac{1}{2!} A^2 h^2 + \cdots\right)$$

$$= e^{At} A = A e^{At}.$$

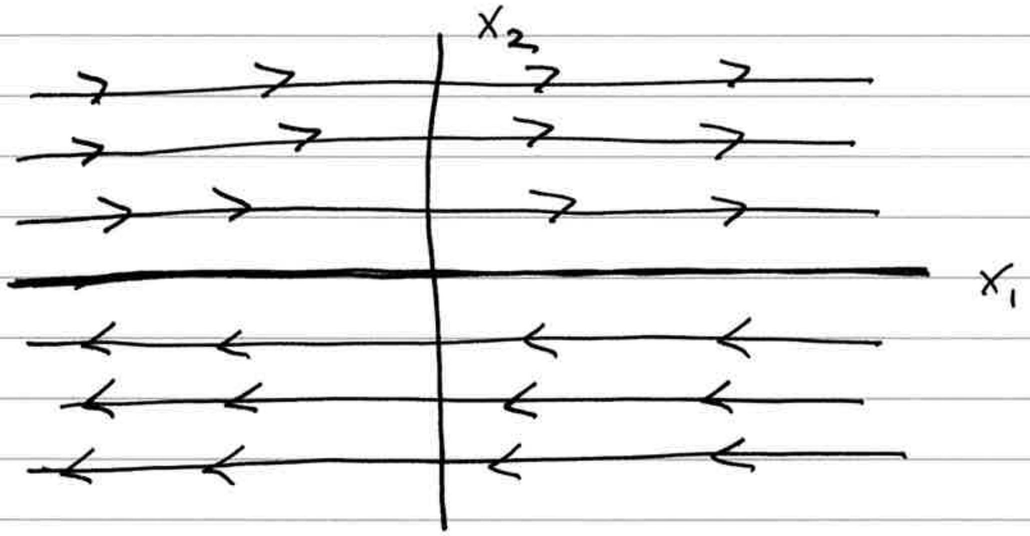
4.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(x_1(0), x_2(0)) = (x_{10}, x_{20})$$

or,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned} \implies \begin{aligned} x_1(t) &= x_{20}\, t + x_{10} \\ x_2(t) &= x_{20} \end{aligned}$$



Every horizontal line is an invariant set.

The x_1 axis is a line of fixed points.

$(x_1, x_2) = (0, 0)$ is unstable.

5)
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(x_1(0), x_2(0)) = (x_{10}, x_{20})$$

$$x_1(t) = x_{10}$$
$$x_2(t) = x_{20}$$

All points in the plane are fixed points.

The origin is Lyapunov Stable.