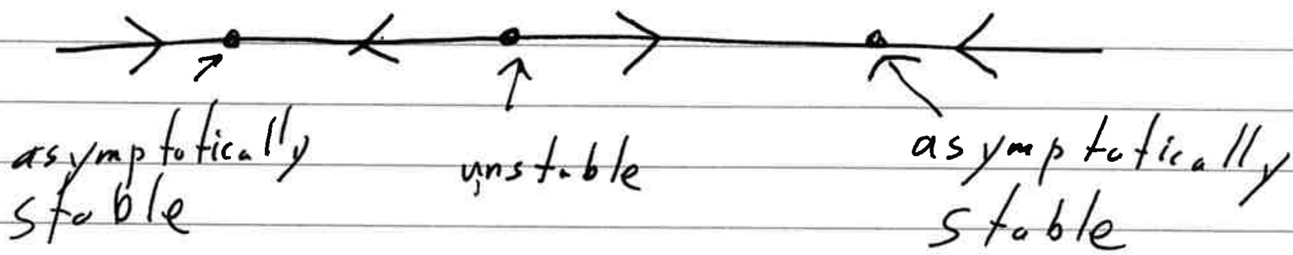
Chapter 3

1. $$\dot{x} = x - x^3$$

equilibria $\quad x = 0, \pm 1$



Problem Set 3.

Q1 (Part 2): Consider the following autonomous vector field on $\mathbb{R}$:

$$\dot{x} = x - x^3, \qquad x \in \mathbb{R}. \tag{1}$$

Compute the flow generated by the vector field and verify the stability for the equilibria directly from the flow.

Solution: We have to find $x(t)$ function. The equation can be solved by the separation of variables:

$$\int \frac{dx}{x - x^3} = \int dt.$$

Since

$$\frac{1}{x - x^3} = \frac{1}{x} + \frac{1}{2}\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x}$$

after integration of both sides we obtain

$$\ln \frac{|x|}{\sqrt{|1-x^2|}} = t + c. \tag{2}$$

Note that square root in eq.(2) is positive quantity. The last equation can be presented in terms of exponents:

$$\frac{x^2}{|1-x^2|} = e^{2c}e^{2t}; \quad \text{or} \quad \frac{x^2}{|1-x^2|} = Ae^{2t}, \quad \text{where} \quad A = e^{2c} > 0.$$

If we set initial conditions as $x(0) = x_0$ then

$$A = \frac{x_0^2}{|1-x_0^2|}. \tag{3}$$

Substitution of $A(x_0)$ gives:

$$\frac{x^2}{|1-x^2|} = \frac{x_0^2}{|1-x_0^2|}e^{2t}. \tag{4}$$

It is the equation we need to solve for x. Of course we have to consider the different intervals on real number line. From the first part of the problem we know that the vector field has equilibria at $x = -1, x = 0$ and $x = 1$. $x = \pm 1$ are asymptotically stable points while $x = 0$ is unstable point of equilibria. These points introduce four intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$. If according to initial condition x_0 is in some of these intervals then $x(t)$ stays in the same interval all time, i.e. if $x_0 \in (-\infty, 0)$ then $x(t) \in (-\infty, 0)$. The same applies to other three intervals as well.

Now let's consider the following cases:

(a) $1-x_0^2<0$ which means that $1-x^2(t)<0$. The case covers intervals (I) $(-\infty,-1)$ and (II) $(1,\infty)$.

(b) $1-x_0^2>$ implies that $1-x(t)>0$ and corresponds to (III) $(-1,0)$ and (IV) $(0,1)$.

In both cases eq.(4) is writtrn as:

$$\frac{x^2}{1-x^2}=\frac{x_0^2}{1-x_0^2}e^{2t} \tag{5}$$

and the solution is

$$x(t)=\frac{x_0e^t}{\sqrt{(1-x_0^2)+x_0^2e^{2t}}}. \tag{6}$$

In case (b) the expression under the square root never vanishes and it is always positive. Hence, the interval of independent variable t doesn't depend on initial condition and $t\in(-\infty,+\infty)$

In case (a), since $1-x_0^2<0$, the expression under the square root may vanish and become negative for some t. The solution is valid when $(1-x_0^2)+x_0^2e^{2t}>0$ or for time interval $t\in(t_0,\infty)$ where

$$t_0=\frac{1}{2}\ln\frac{x_0^2-1}{x_0^2}.$$

To verify the stability of equilibia we have to compute $\lim_{t\to\infty}x(t)$ in each interval. If we take into account that

$$\lim_{t\to\infty}\sqrt{(1-x_0^2)+x_0^2e^{2t}}=|x_0|e^t$$

for the limits of $x(t)$ is obtained

$$\lim_{t\to\infty}x(t)=-1 \quad \text{in intervals (I) and (III)},$$

$$\lim_{t\to\infty}x(t)=1 \quad \text{in intervals (II) and (IV)}.$$

It means that the points of equalibria $x=\pm1$ are asymptotically stable. As we see no solution converges to the point of equilibria $x=0$ in intervals (III) and (IV) when $t\to\infty$.

Also we can calculate the limits at $-\infty$ in intervals (III) and (IV):

$$\lim_{t\to-\infty}x(t)=0$$

and in intervals (I) and (II) the limits at t_0

$$\lim_{t\to t_0}x(t)=-\infty \quad \text{in interval (I)},$$

$$\lim_{t\to t_0}x(t)=\infty \quad \text{in interval (II)}.$$

2. Proof by contradiction.

Suppose $p \notin M$, but there exist T such that $\phi_T(p) \in M$. Then, since M is invariant

$$\phi_t(\phi_T(p)) \in M \quad \forall t.$$

However, take $t = -T$. Then we have $p \in M$, which is a contradiction.

3. First, note that the x and y components of the vector field are independent.

The equilibria (0 dimensional invariant sets) are

$(0, 0)$, $(0, 1)$, $(0, -1)$

Note that for $y = 0$, $\dot{y} = 0$. This implies that the x axis is a 1-dimensional invariant set

At $x = 0, \pm 1$ $\dot{x} = 0$, and this implies that

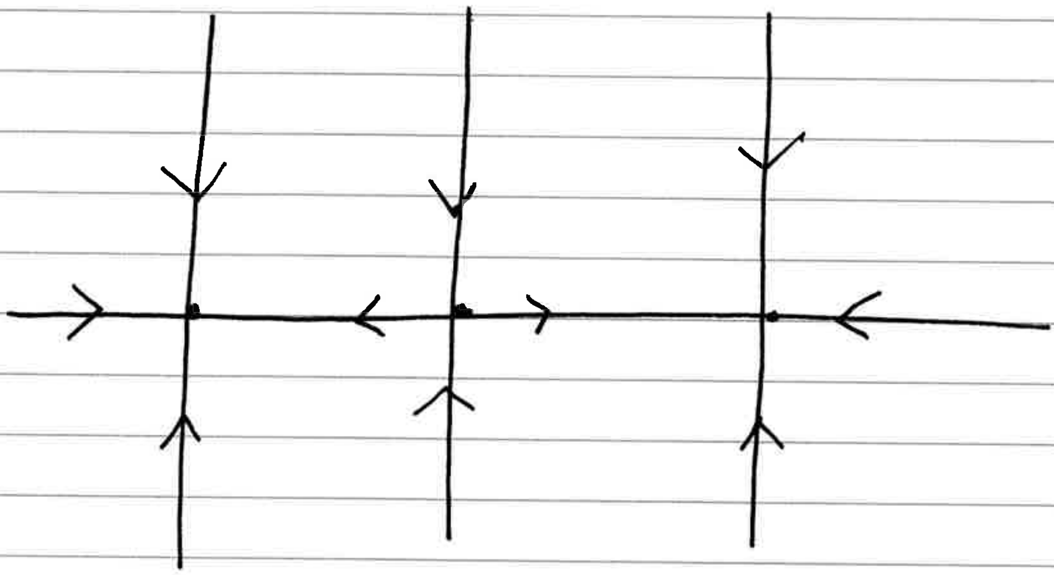
$\{ (x, y) \mid x = 0, -\infty < y < \infty \}$

$\{ x, y) \mid x = 1, -\infty < y < \infty \}$

$\{ x, y) \mid x = -1, -\infty < y < \infty \}$

are all invariant lines, i.e. one dimensional invariant sets

So far, the phase portrait appears as below



Now we use problem 2, i.e. no trajectory can cross an invariant set. This implies that the phase plane is divided into 8 invariant regions.

$\{(x, y) \mid -\infty < x < 1, \quad 0 < y < \infty\}$
$\{(x, y) \mid -\infty < x < 1, \quad -\infty < y < 0\}$
$\{(x, y) \mid -1 < x < 0, \quad 0 < y < \infty\}$
$\{(x, y) \mid -1 < x < 0, \quad -\infty < y < 0\}$
$\{(x, y) \mid 0 < x < 1, \quad 0 < y < \infty\}$
$\{(x, y) \mid 0 < x < 1, \quad -\infty < y < 0\}$
$\{(x, y) \mid 1 < x < \infty, \quad 0 < y < \infty\}$
$\{(x, y) \mid 1 < x < \infty, \quad -\infty < y < 0\}$

Attracting set $(-1, 0)$ – basin of attraction $x < 0$ (left half plane)

Attracting set $(0, 1)$ – basin of attraction $x > 0$ (right half plane).

Two heteroclinic orbits

$\{(x, y) \mid y = 0, \ -1 < x < 0\}$

$\{(x, y) \mid y = 0, \ 0 < x < 1\}$

There can be no (non-equilibrium) periodic orbits since $y(t) = y_0 e^{-t}$.