UNIVERSITY OF BRISTOL

School of Mathematics

Ordinary Differential Equations 2

Test

Solution> <Solution

20 November 2019

40 minutes

This paper contains **three** questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Comment>

Marking Scheme:

Fully correct answer: 2 marks.

Answer with some relevant correct element: 1 mark.

Answer with no relevant correct element: 0 marks.

<Comment

1. (**2 marks**)

   Can an autonomous (one-dimensional) ODE on $\mathbb{R}$ have a homoclinic orbit? If yes, give an example, if no, explain why not.

   Solution>
   No, for the same reason that autonomous ODEs on $\mathbb{R}$ do not have periodic orbits: A homoclinic, like a periodic orbit would require the solution to be both increasing and decreasing at the same point, which violates autonomy.
   Note: Some students have presented the ODE $\dot{x} = 0$ as an example and justified this (in class, after the test) by arguing that by the definition in the text, every equilibrium is a homoclinic orbit. Although this is not what the text intended to say, full marks should be given for this example and similar examples. <Solution

2. (**2 marks**)

   Give an example of an autonomous (one-dimensional) ODE on $\mathbb{R}$ which has non-equilibrium orbits, all of which are heteroclinics.

   Solution>
   Since we want non-equilibrium orbits and want all such orbits to be heteroclinics, we need a set of equilibria that is unbounded, both on the positive reals and on the negative reals. The easiest example is to choose these to be zeros of a periodic function which changes sign, e.g., sin. Thus we have the example $\dot{x} = \sin(x)$. (Other answers are possible, of course.)
   Note: Full marks to be given even if the ODE presented as example has issues with existence or uniqueness of solutions, for example,

   $$\dot{x} = \begin{cases} 0 \text{ if } x \in \mathbb{Z}, \\ 1 \text{ otherwise.} \end{cases}$$

   <Solution

3. (**6 marks**) Consider the Linear ODE on $\mathbb{R}^2$,

   $$\dot{x} = Ax.$$

   Here, $A \in \mathbb{R}^{2\times 2}$ has $(1, 1)$ as an eigenvector with eigenvalue 1, and also $(1, -1)$ as an eigenvector with eigenvalue -1.

   (a) (**2 mark**) Characterise the stability of $(0, 0)$.

   (b) (**2 marks**) Draw a phase diagram, clearly indicating the invariant subspaces and their stability.

   (c) (**2 marks**) Write down the flow. You may leave you answer as a product of matrices (i.e., there is no need to simplify your answer).

   Solution>

   (a) Saddle since A has eigenvalues of opposite sign.

   (b) The phase diagram is as for a saddle (e.g., Figure 5.6 on page 55 of the textbook) with stable subspace along $(1, -1)$ and unstable subspace along $(1, 1)$.

(c)

$$\phi_t(x_0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} x_0,$$

or, equivalently,

$$\phi_t(x_0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} x_0.$$

<Solution

End of examination.