UNIVERSITY OF BRISTOL

School of Mathematics

Ordinary Differential Equations 2

MATH20101

(Paper code MATH–20101J)

January 2019 2 hours 30 minutes

This paper contains **four** questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Comment>

Annotations:

Exercise: Very similar to material seen in lectures or homework. (**29 marks**)

Standard: Unseen but straightforward. (**44 marks**)

Hard: Unseen and hard. (**27 marks**)

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<Comment

1. (**25 marks**)

This question consists of two independent parts.

(a) (**12 marks**)
Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = \lambda y,$$
$$\dot{y} = \lambda x,$$

where $\lambda > 0$.

Comment>

Students have seen this ODE (Question 6 of Chapter 2, page 30) as a non-set homework problem.

<Comment

i. (**8 marks**)
Show that the resulting flow is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Reminder: $\cosh(\lambda t) = \frac{1}{2}\left(e^{\lambda t} + e^{-\lambda t}\right)$ *and* $\sinh(\lambda t) = \frac{1}{2}\left(e^{\lambda t} - e^{-\lambda t}\right)$.

Solution> (**Exercise**)

The ODE can we written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (\textbf{1 mark}).$$

Thus the eigenvalues are $\pm\lambda$ (**1 mark**) and the corresponding eigenvectors are $\frac{1}{\sqrt{2}}(1, \pm 1)$ (**2 marks**). Thus

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^T \quad (\textbf{2 marks})$$

and

$$e^{\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} t} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^T = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix} \quad (\textbf{2 marks}).$$

<Solution

ii. (**4 marks**)
Compute the global stable and unstable manifolds of the equilibrium.

Solution> (**Standard**)

It is clear from the calculation above that the global stable manifold is $\{(x, -x) \mid x \in \mathbb{R}\}$ (**2 marks**) and the global unstable manifold is $\{(x, x) \mid x \in \mathbb{R}\}$ (**2 marks**).

<Solution

(b) (**13 marks**)

Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = -y^2,$$
$$\dot{y} = -y.$$

Compute the stable and centre manifolds of the origin.

Comment> This is a slight modification of a question from the 2017-18 exam, which exam and solutions are available to the students.

<Comment

Solution> (**Standard**)

Solving the ODE for y we obtain

$$y(t) = y(0)e^{-t} \quad \textbf{(1 mark)}$$

and thus

$$x(t) = \left(x(0) - \frac{1}{2}(y(0))^2\right) + \frac{1}{2}(y(0))^2 e^{-2t} \quad \textbf{(2 marks)}.$$

We deduce that $x = \frac{1}{2}y^2$ is the stable manifold at the origin (**5 marks**). It also follows that the centre manifold is $y = 0$, i.e. the x-axis (**5 marks**).

<Solution

2. (**25 marks**)

Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = y,$$
$$\dot{y} = x - x^3.$$

(a) (**4 marks**)

Compute the stability of the equilibrium at $(0, 0)$.

Solution> (**Exercise**)

The Jacobian at $(0, 0)$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (**1 mark**) and thus the eigenvalues are ± 1 (**1 mark**). Thus the linear stabiltiy is that of a saddle (**1 mark**); from hyperbolicity, the same is true for the nonlinear system too (**1 mark**).

<Solution

(b) (**2 marks**)

Show that the function

$$V(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

is constant along trajectories.

Solution> (**Exercise**)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \quad \textbf{(1mark)} \tag{1}$$
$$= y\dot{y} - x\dot{x} + x^3\dot{x} = y(x - x^3) - xy + x^3 y = 0 \quad \textbf{(1mark)}. \tag{2}$$

<Solution

(c) (**7 marks**)

Sketch the phase portrait.

Solution> (**Standard**)

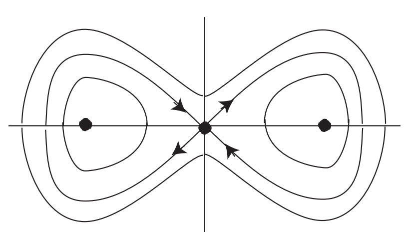

Figure 1: The phase portrait.

<Solution

(d) (**6 marks**)

Are there any homoclinic orbits? If yes, how many? Indicate them on the phase portrait. What value(s) does V take on them?

Solution> (**Standard**)

From the phase portrait it is clear that there are two homoclines (**2 marks**); these are indicated with arrows in Figure 1 (**2 marks**). $V = 0$ on a homocline (**2 marks**).

<Solution

(e) (**3 marks**)

Are there any periodic orbits, other than the equilibrium solutions?

Solution> (**Standard**)

Yes, there are non-equilibrium periodic orbits: the homoclinic orbits divide the plane into three distinct families of periodic orbits, and each family contains an infinite number of periodic orbits. (**3 marks**).

<Solution

(f) (**3 marks**)

Are there any heteroclinic orbits?

Solution> (**Standard**)

No, this is clear from the phase portrait.

<Solution

3. (**25 marks**)

Although this question is independent of Question 2, it is recommended that Question 2 be attempted before this question.

Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - \alpha y, \end{aligned}$$

where $\alpha = 1$.

(a) (**2 mark**)
Identify the equilibria.

Solution> (**Exercise**)

The equilibria are at $(0,0)$, and $(\pm 1, 0)$.

<Solution

(b) (**4 marks**)
Compute the stability of the equilibria other than $(0,0)$.

Solution> (**Exercise**)

The Jacobian at $(\pm 1, 0)$ is $\begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$ (**1 mark**). Thus, the trace, which is the sum of the eigenvalues, is negative and the determinant, which is the product of the eigenvalues, is positive. It follows that the eigenvalues are both negative (**1 mark**). Thus both equilibria have linear stability of sinks (**1 mark**); from hyperbolicity, the same is true for the nonlinear system too (**1 mark**).

<Solution

(c) (**2 marks**)
Are there any non-equilibrium periodic orbits?

Solution> (**Exercise**)

The divergence of the vector field is -1. From Bendixon's criterion there are no non-equilibrium periodic orbits.

<Solution

(d) (**15 marks**)
Use the LaSalle Invariance Principle to show that the trajectory starting at any $(x_0, y_0) \in \mathbb{R}^2$ approaches one of the equilibria as $t \to \infty$.
Hint: Use $V(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ *as a Lyapunov function.*

Solution> (**Hard**)

Let $\mathcal{M}$ be the set bounded by the contour $V(x, y) = c$, where c is chosen sufficiently large so that M contains the three equilibria. This is a positive invariant set since, by a calculation similar to (2), $\dot{V} = -y^2$ on the boundary of $\mathcal{M}$. (**5 marks**)

Let

$$E \equiv \left\{ (x, y) \in \mathcal{M} \mid \mid \dot{V}(x, y) = 0 \right\}.$$

Since $\dot{V} = -y^2$, E is the intersection of the x-axis with $\mathcal{M}$. (**3 marks**)

Now

$$M = \{\text{The union of all trajectories that start in E and remain in E for all } t \geqslant 0.\}$$

Hence M consists of the three equilibrium points. Therefore, by the LaSalle Invariance principle the trajectory through any initial condition in $\mathcal{M}$ must approach M as $t \to \infty$ (**7 marks**).

<Solution

(e) (**2 marks**)

For which values of $\alpha \in \mathbb{R}$ do all answers to questions (3a) to (3d) above remain unchanged?

Solution> (**Standard**)

It is clear that it is necessary and sufficient that $\alpha > 0$.

<Solution

4. (**25 marks**)

Consider the ODE on $\mathbb{R}^2$ which in *polar coordinates* is given by

$$\dot{r} = r(\mu - r^2),$$
$$\dot{\theta} = 2\sin^2\frac{\theta}{2},$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) (**7 marks**)
Describe the bifurcations of the equilibria as μ is varied. What would be an appropriate name for this bifurcation?
Solution> (**Exercise**)
The set of equilibria change from $\{(0,0)\}$ when $\mu \leqslant 0$ to $\{(0,0),(\sqrt{\mu},0)\}$ when $\mu > 0$ (**4 marks**). "Pitchfork", but one prong of the 'fork' is missing since $r \geqslant 0$ (**3 marks**).
<Solution

(b) (**18 marks**)
Let $\mu = 1$ and $\phi_t(\cdot), t \in \mathbb{R}$ be the resulting flow.

i. (**6 marks**)
Without computing the divergence of the vector field, show that there exists a homoclinic orbit but no non-equilibrium periodic orbits.
Solution> (**Standard**)
We see that $\dot{r}$ is monotone in $(0,1)$ and $(1,\infty)$. Thus periodic orbits are possible only for $r = 0$, which is an equilibrium, or $r = 1$ (**3 marks**).
Let $r = 1$. Again we see that $\dot{\theta}$ is non-negative but vanishes at $\theta = 0$. Thus the unit circle without $(1,0)$ is a homoclinic but there are no periodic non-equilibrium orbits (**3 marks**).
<Solution

ii. (**6 marks**)
Let (r_o, θ_o) be a point other than the origin. Show that

$$\lim_{t\to\infty} \phi_t(r_o, \theta_o) = (1,0). \qquad (3)$$

Solution> (**Hard**)
First, note that $(r,\theta) = (1,0)$ is an equilibrium (**1 mark**). Next, since, by hypothesis, $r_o \neq 0$, from the ODE it follows that $r \to 1$ as $t \to \infty$ (**2 marks**). Finally, $\dot{\theta} \geqslant 0$, with $\dot{\theta} = 0$ only at $\theta = 0$ so $\theta \to 0$ (anticlockwise) as $t \to \infty$ (**2 marks**). It follows that all trajectories (other than the origin) spiral anticlockwise to $(1,0)$ (**1 marks**).
<Solution

iii. (**6 marks**)
Is $(1,0)$ a stable equilibrium? Explain why your answer is consistent with Equation (3) above.
Solution> (**Hard**)
No, since for every $\epsilon > 0$, the point $(r,\theta) = (1,\epsilon)$ is carried out of arbitrary small neighbourhoods of $(1,0)$ (**4 marks**). This is consistent with (3) since stability is

concerned with the behaviour of solutions for all $t > 0$, and not only in the limit $t \to \infty$ (**2 marks**).

<Solution

End of examination.