Solutions to ODE2 Exam
January, 2018

1. $$\dot{x} = -x$$
$$\dot{y} = -x^2$$

a) (5 marks) $\underline{x(t) = x_0 e^{-t}}$

$$\int_{y(0)}^{y(t)} dy' = -x_0^2 \int_0^t e^{-2t'} dt'$$

$$\Rightarrow \quad y(t) = y_0 - \frac{x_0^2}{2}\left(1 - e^{-2t}\right)$$

b) (5 marks) The vector field $\underline{\text{cannot}}$ have periodic orbits since

$$x(t) = x_0 e^{-t}$$

c) (5 marks) Center manifold of the origin, $x = 0$

d) (5 marks) Stable manifold of the origin

By inspecting $y(t)$,

$$y = \frac{x^2}{2}$$

d) (5 marks) The origin is Lyapunov stable. From the expression for the flow, for <u>any</u> neighbourhood of the origin, trajectories with initial conditions in that neighbourhood remain in the neighbourhood for as $t \to \infty$.

2.

$$\dot{x} = y$$
$$\dot{y} = -x - x^2 y + z x y$$
$$\dot{z} = -z + x z^2$$

a) (5 marks) Matrix associated with linearization about the origin

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

eigenvalues $-1, \underbrace{\pm i}_{\text{not hyperbolic}}$

b) (5 marks) Stable manifold of the origin

$$x = y = 0 \qquad (z \text{ axis})$$

c) (5 marks) Center manifold of the origin

$$z = 0 \qquad (x-y \text{ plane})$$

d) (10 mark) Vector field restricted to the Center manifold.

$$\dot{x} = y$$
$$\dot{y} = -x - x^2 y$$

Apply La Salle invariance principle

Choose $V = \frac{1}{2}(x^2 + y^2)$

$$\dot{V} = x\dot{x} + y\dot{y}$$
$$= xy + y(-x - x^2 y) = -x^2 y^2$$

Consider

$$\frac{1}{2}(x^2 + y^2) = C$$

For C large enough, this bounds a positive invariant set containing the origin.

$\dot{V} = 0$ on $x=0$ and $y=0$

The set of points that start in this set and remain in this set for all positive time is the origin, $x=y=0$.

Therefor by the La Salle invariance principle, trajectories through all points approach the origin as $t \to \infty \Rightarrow$ asymptotic stability.

3. $\dot{x} = \mu x - \omega y - \frac{x}{2}\left(x^2+y^2 - \frac{2}{3}(x^2+y^2)^{3/2}\right)$

$\dot{y} = \omega x + \mu y - \frac{y}{2}\left(x^2+y^2 - \frac{2}{3}(x^2+y^2)^{3/2}\right)$

a) (5 mark) At $\mu = 0$ the matrix associated with linearization about the origin is

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \Rightarrow \text{eigenvalues } \underbrace{\pm i\omega}_{\text{nonhyperbolic}}$$

b) (5 marks) In polar coordinates

$$\dot{r} = \mu r - \frac{r^3}{2} + \frac{r^4}{3}$$

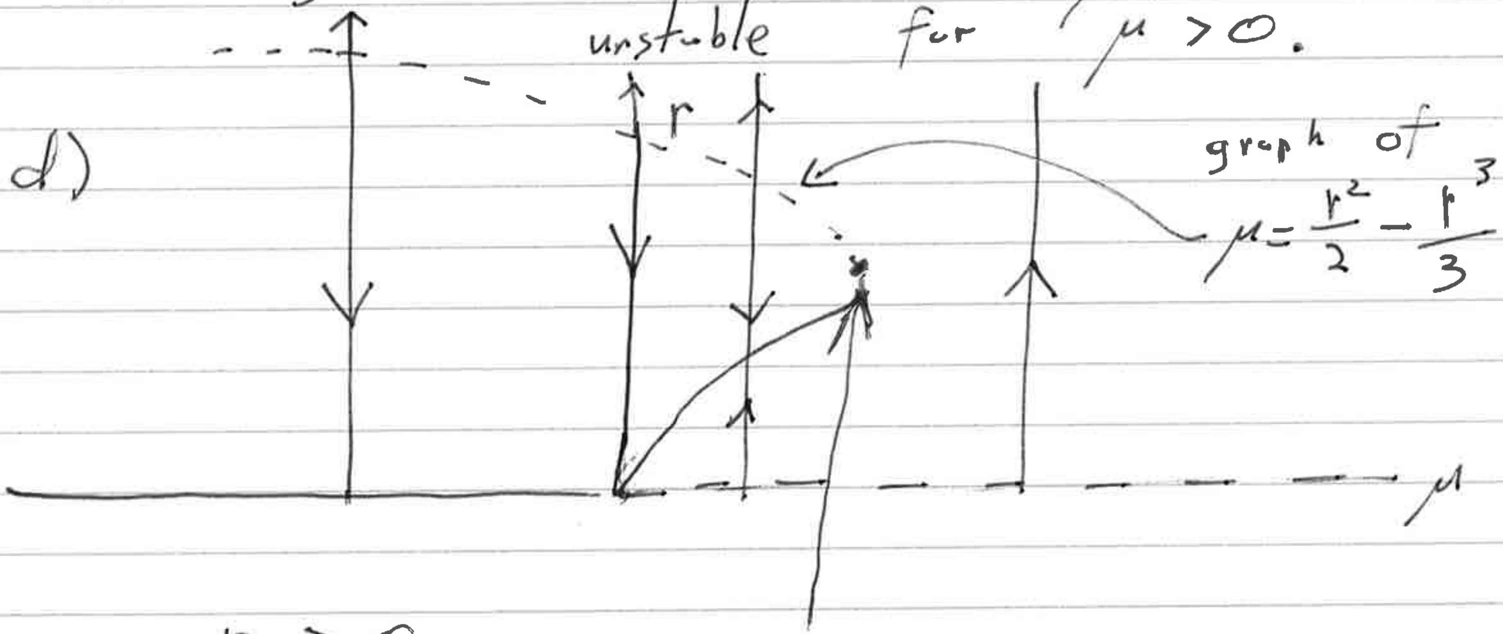
$$\dot{\theta} = \omega$$

c) (5 marks) From

$$\dot{r} = \mu r - \frac{r^3}{2} + \frac{r^4}{3}$$

The origin is stable for $\mu < 0$
unstable for $\mu > 0$.

d)



$\underline{r \geq 0}$

saddle-node bifurcation
of periodic orbits

4) $\dot{x} = -x$
$\dot{y} = \sin y$

$x \in \mathbb{R}, \quad -\pi \le y \le \pi$

a) Fixed Points (5 marks)

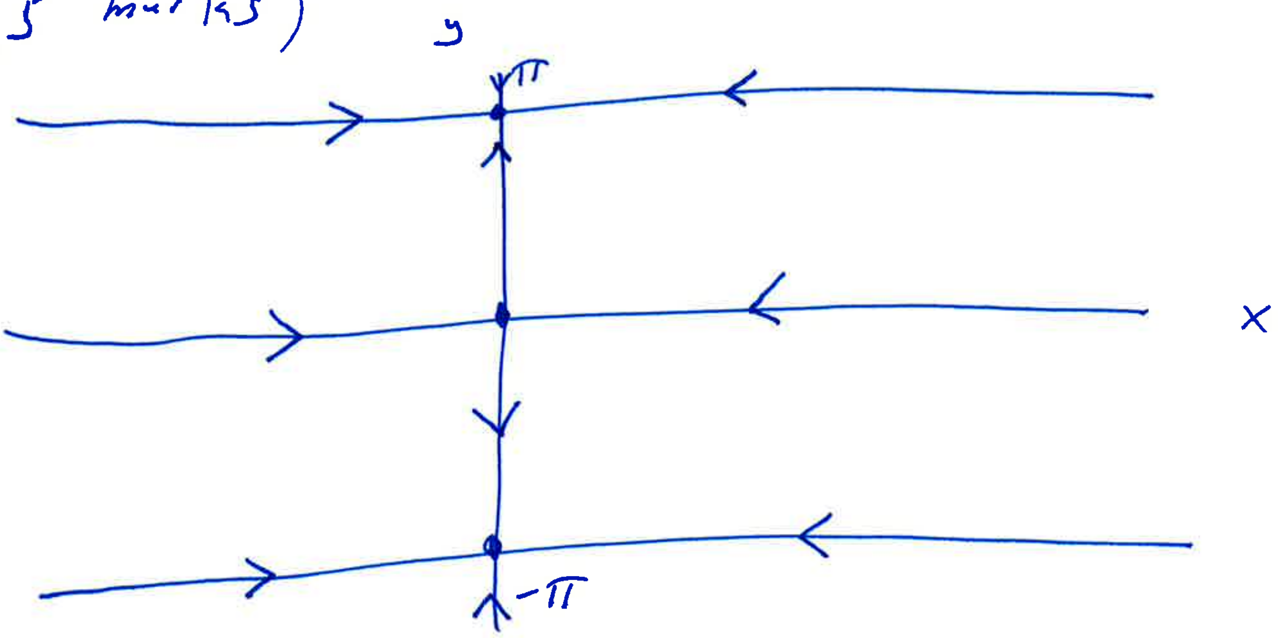
b) Matrix associated with linearization (5 marks)

$(0, \pi)$ $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ sink

$(0, 0)$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ saddle

$(0, -\pi)$ $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ sink

c) (5 marks)



d) (5 marks)

global stable manifold. $x \in \mathbb{R}$, $y = 0$

global unstable manifold, $x = 0$, $-\pi < y < \pi$

e) (5 marks)
There are no non hyperbolic fixed points.