ODE2 : Problem Set 2 Solutions

1. No. Let p denote the hyperbolic fixed point, let x denote any point on the homoclinic orbit $(x \neq p)$, and let $\phi_t(\cdot)$ denote the flow generated by the vector field. Suppose there exists a (finite) T such that $\phi_T(x) = p$. Then $x = \phi_{-T}(p) = p$ (since p is an equilibrium point. This is a contradiction that arises from the assumption that there is a finite T such that $\phi_T(x) = p$.

2. No. Consider the vector field:

$$\dot{x} = f(x),\; x \in \mathbb{R}.$$

   If $f(x) \neq 0$ for any x. Then either $f(x) > 0$ or $f(x) < 0$, for all x. Hence $\dot{x}$ is always increasing, or always decreasing. In either case, periodic orbits are not possible.

3. Yes. Consider the example $\dot{x} = \cos t$. A solution is $x(t) = \sin t$, which is periodic in time.

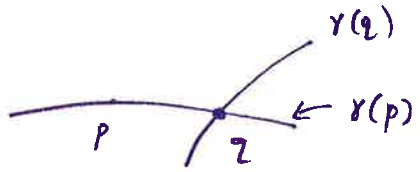
4. Yes. Consider the vector field:

$$\dot{\theta} = f(\theta),\; x \in S^1.$$

   Since the vector field has no equilibria the same argument in (a) implies that $\dot{\theta}$ is either always increasing or always decreasing.In either case, because the vector field is defined on a circle, *all* orbits are periodic.

2.7 · Orbits either coincide or are disjoint — i.e., orbits cannot 'cross'.

proof:



Let $\underbrace{q \in \cancel{\varphi}\, \gamma(p)}_{\exists \tau \text{ s.t. } \varphi_\tau(p) = q}$. Then

$$\begin{aligned} \gamma(q) &= \{ \varphi_t(q) \mid t \in \mathbb{R} \} \\ &= \{ \varphi_t(\varphi_\tau(p)) \mid t \in \mathbb{R} \} \\ &= \{ \varphi_{t+\tau}(p) \mid t \in \mathbb{R} \} = \gamma(p) \end{aligned}$$