1a) $\ddot{\theta} + \delta \dot{\theta} + \sin\theta = F\cos\omega t$

$$\dot{\theta} = v$$
$$\dot{v} = -\delta v - \sin\theta + F\cos\omega t$$

dependent Variables : $(\theta, v) \in S^1 \times \mathbb{R}^1$

independent Variable : t

parameters : δ, F, ω

nonlinear, nonautonomous

1b) $\ddot{\theta} + \delta \dot{\theta} + \theta = F \cos \omega t$

$$\dot{\theta} = v$$
$$\dot{v} = -\delta v - \theta + F \cos \omega t$$

dependent variables: $(\theta, v) \in S^1 \times \mathbb{R}^1$

independent variable: t

parameters: δ, F, ω

linear, nonautonomous

1c) $$\frac{d^3y}{dx^3} + x^2 y \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = V$$

$$\frac{d^2y}{dx^2} = u = \frac{dv}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{du}{dx}$$

then,
$$\frac{dy}{dx} = V$$

$$\frac{dV}{dx} = u$$

$$\frac{du}{dx} = -x^2 yV - y$$

dependent Variables : $(y, v, u) \in \mathbb{R}^3$

independent Variable : x

no parameters

nonlinear, nonautonomous

1d) $\ddot{x} + \delta \dot{x} + x - x^3 = \Theta$

$\ddot{\theta} + \sin\theta = 0$

$$\dot{x} = v$$
$$\dot{v} = -\delta v - x + x^3 \cancel{=} \Theta$$

$$\dot{\theta} = w$$
$$\dot{w} = -\sin\theta$$

dependent Variables: $(x, v, \theta, w) \in \mathbb{R}^1 \times \mathbb{R}^1 \times S^1 \times \mathbb{R}^1$

independent Variable: t

parameters: δ

nonlinear, autonomous

1e) $\ddot{\theta} + \delta \dot{\theta} + \sin\theta = x$

$\ddot{x} - x + x^3 = 0$

$\dot{\theta} = v$

$\dot{v} = -\delta v - \sin\theta + x$

$\dot{x} = y$

$\dot{y} = x - x^3$

dependent Variable : $(\theta, v, x, y) \in S^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$

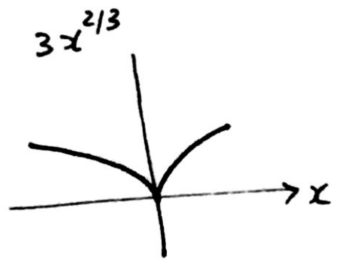
independent Variable : t

parameter : δ

nonlinear, autonomous

Q2 The function $f(x) = 3x^{2/3}$ is C^1 except at 0

(Google "x^(2/3)" and "x^(-1/3)" to see plots of this function and its derivative



This is a sufficient condition for a unique solution to exist in $(-\infty, 0)$ and $(0, \infty)$.

[See page 21.]

Q3 Consider the vector field:

$$\dot{x} = -x + x^2 \quad , x(0) = x_0 \quad , x \in \mathbb{R}$$

Determine the time interval of existence of all solutions as a function of the initial conditions.

Since $\dot{x} = x(x-1)$ we note firstly that $x(t) = 0, \forall t$ and $x(t) = 1, \forall t$ are solutions with initial conditions $x_0 = 0$ and $x_0 = 1$ respectively. Also as this ODE is autonomous — so that trajectories cannot "intersect" we know that all other solutions $x(t)$ must, for all t over the time interval of existence of $x(t)$, lie in precisely one of the following intervals: $(-\infty, 0)$, $(0, 1)$ or $(1, \infty)$. $(*)$ [So for all such solutions $x(t) \neq 0$ whenever $x(t)$ is defined. Also $x_0 \in \mathbb{R} \setminus \{0,1\}$.]

So now suppose that $x_0 \notin \mathbb{R} \setminus \{0,1\}$. Then we can safely separate variables to obtain:

$$\int_{x_0}^{x} \frac{dz}{z(z-1)} = \int_0^t ds$$

i.e.

$$\int_{x_0}^{x} \frac{1}{z-1} - \frac{1}{z} \, dz = \int_0^t ds$$

So integrating we get: $\ln\left|\frac{x-1}{x}\right| - \ln\left|\frac{x_0-1}{x_0}\right| = t$

and now exponentiating and rearranging: $\left|\frac{x-1}{x}\right| = \left|\frac{x_0-1}{x_0}\right| e^t \quad (**)$

However, by $(*)$ (and the fact that $e^t > 0$ for all t), we know that $(**)$ is equivalent to:

$$\frac{x-1}{x} = \frac{x_0-1}{x_0} e^t$$

And, rearranging we find: $x(t) = \dfrac{x_0}{(1-x_0)e^t + x_0}$

Now $x(t)$ "explodes" when $(1-x_0)e^t + x_0 = 0$, i.e. when

$$t(x_0) = \ln\left(\frac{x_0}{x_0-1}\right)$$

Here we have 3 cases:

<u>Case 1</u> $x_0 \in (-\infty, 0)$. Then $x_0/(1-x_0) \in (0,1)$ and so $t(x_0)$ is defined and $t(x_0) < 0$.

<u>Case 2</u> $x_0 \in (0,1)$. Then $x_0/(1-x_0) \in (-\infty, 0)$ and so $t(x_0)$ is not defined. (i.e. $x(t)$ is defined for all t.)

<u>Case 3</u> $x_0 \in (1, +\infty)$. Then $x_0/(1-x_0) \in (0,\infty)$ and so $t(x_0)$ is defined and $t(x_0) > 0$.

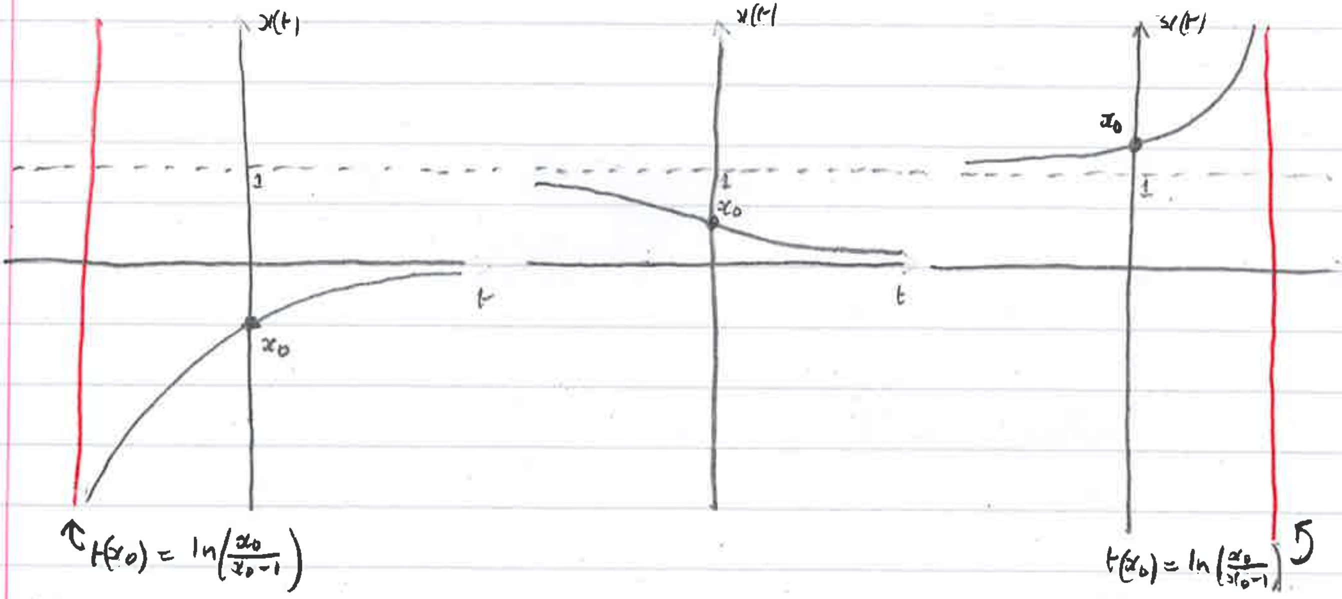
We conclude that:

$x_0 < 0 \Rightarrow$ the time interval of existence of $x(t)$ is $\left(\ln\left(\frac{x_0}{x_0-1}\right), +\infty\right)$

$0 \le x_0 \le 1 \Rightarrow$ the time interval of existence of $x(t)$ is $(-\infty, +\infty)$

$x_0 > 1 \Rightarrow$ the time interval of existence of $x(t)$ is $\left(-\infty, \ln\left(\frac{x_0}{x_0-1}\right)\right)$

Further analyses shows that the trajectories are as follows:



$x_0 \in (-\infty, 0)$ $\qquad$ $x_0 \in (0,1)$ $\qquad$ $x_0 \in (1, +\infty)$

Charles (based on original solution by Yani).

The ODE can we written as $\dot{x} - a(t)x = b(t)$ (*)

Q4

Let $A(t) = -\int_0^t a(\tau)\,d\tau$ so $A'(t) = -a(t)$.

Then multiplying both sides of (*) by $e^{A(t)}$ we obtain:

$$\frac{d}{dt}\left(e^{A(t)}x(t)\right) = e^{A(t)}b(t)$$

Thus

$$x(t) = e^{-A(t)}\left(\int_0^t e^{A(\tau)}b(\tau)\,d\tau + c\right) \qquad (**)$$

where c is a constant of integration.

For (**) to hold we need the following ~~two~~ two integrals to exist $\forall\, t \in \mathbb{R}$

- $\int_0^t a(\tau)\,d\tau = -A(t)$
- $\int_0^t e^{A(\tau)}b(\tau)\,d\tau$

–

Note that an alternate sufficient condition can be written, directly from (*):

From page 21 it is sufficient that

$$f(x,t) := a(t)x + b(t)$$

be C^1. For this it suffices that $a(t)$ and $b(t)$ be C^1 functions.