UNIVERSITY OF BRISTOL

School of Mathematics

Ordinary Differential Equations 2

Sample Test

Solution>	<Solution

November 2019

40 minutes, but designed to be completed in 20 minutes

This paper contains **two** questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. (**5 marks**)

   Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = x\ f(x, y),$$
$$\dot{y} = y\ g(x, y),$$

   where $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ are smooth functions.

   Show that each of the four quadrants, i.e. each of the sets

$$\left\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\right\},$$
$$\left\{(x, y) \in \mathbb{R}^2 \mid x > 0, y < 0\right\},$$
$$\left\{(x, y) \in \mathbb{R}^2 \mid x < 0, y > 0\right\},$$
$$\left\{(x, y) \in \mathbb{R}^2 \mid x < 0, y < 0\right\},$$

   is an invariant set.

   Solution>

   Since $\dot{x} = 0$ when $x = 0$, no trajectory can cross the y-axis. Similarly, since $\dot{y} = 0$ when $y = 0$, no trajectory can cross the x-axis. It follows that each quadrant is an invariant set.

   <Solution

2. (**10 marks**)

Which of the following trajectories on $\mathbb{R}^2$ can be solutions of a first-order autonomous linear ODE?

(a) $(x(t), y(t)) = (3e^t + e^{-t}, e^{2t})$.
(b) $(x(t), y(t)) = (3e^t + e^{-t}, e^t)$.
(c) $(x(t), y(t)) = (3e^t + e^{-t}, te^t)$.
(d) $(x(t), y(t)) = (3e^t, t^2e^t)$.
(e) $(x(t), y(t)) = (e^t + 2e^{-t}, e^t + 2e^{-t})$.

You must justify your answer.

Solution>

From the canonical forms we see that the solution of a first-order two-dimensional autonomous linear ODE is one of the following:

- A linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.
- A linear combination of $e^{\lambda t}$ and $te^{\lambda t}$.
- A linear combination of $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$.

This eliminates (2a), (2c) and (2d).

For (2e), we see that $x = y$ so the solution lies in a one-dimensional subspace of $\mathbb{R}^2$. The solution of a first-order one-dimensional autonomous linear ODE must be of the form $e^{\lambda t}$, so (2e) is eliminated as well.

We conclude that only (2b) is the solution of a first-order two-dimensional autonomous linear ODE.

<Solution

End of examination.