Ordinary Differential Equations 2
Pre-Test Homework

Due: 4 pm on Tuesday 12 November 2019

1. (Based on Q1 from 2016-17 exam)

   Consider the following autonomous vector field on $\mathbb{R}^2$:

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy$$

   Give conditions on a, b, c, d for which:

   (a) there are no periodic orbits.

   (b) all orbits are periodic.

   (c) both the x- and y-axes (i.e., the lines $x = 0$ and $y = 0$) are invariant manifolds.

   You do *not* have to give all possible conditions on the constants in order for the dynamical condition to be satisfied. One condition will be sufficient, but you must justify your answer.

2. (Based on Q3 from 2017-18 exam)

   Consider the following autonomous vector field on $\mathbb{R}^2$ with $\omega > 0$ being a parameter:

$$\dot{x} = -\omega y - \frac{x}{2}\left(x^2 + y^2 - \frac{2}{3}(x^2 + y^2)^{\frac{3}{2}}\right),$$
$$\dot{y} = \omega x - \frac{y}{2}\left(x^2 + y^2 - \frac{2}{3}(x^2 + y^2)^{\frac{3}{2}}\right).$$

   (a) Show that $(x, y) = (0, 0)$ is an equilibrium.

   (b) Transform the vector field to polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$.

   Hint: First show that

$$r\dot{r} = x\dot{x} + y\dot{y},$$
$$r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

   and use that to write the original ODE in terms of $r, \theta, \dot{r}, \dot{\theta}$.