MATH20101 Ordinary Differential Equations 2

Isaac V. Chenchiah
https://www.isaacchenchiah.org/ode2.html
Isaac.Chenchiah@bristol.ac.uk

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These notes are intended as a supplement to the textbook for the unit (Stephen Wiggins, Ordinary Differential Equations) and are subject to modification as the term progresses.[1]

Contents

1 Notes

1.1 Chapter 2

The words 'trajectory' and 'orbit' first used on pages 27 and 29 (Questions 1-4), respectively, are not defined in the text:

'Trajectory' is another word for solution. Typically we use the word trajectory for a solution represented as a curve on the phase plane.

Definition 1.1 (Orbit)**.** *Let p be a point on the phase space. The orbit of p is the set*

$$\{\phi_t(p) \mid t \in \mathbb{R}\}$$

where $\phi_t(\cdot)$ is the flow (see pages 24-25).

1.2 Chapter 3

In Definition 16 on page 33 the term 'neighbourhood' means "sufficiently small open set".

1.3 Chapters 4 and 5

Given $A \in \mathbb{R}^{2\times 2}$, *which is not a multiple of the identity*, we identify its canonical form by considering the roots of the characteristic polynomial

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0.$$

Three case arise:

1. Two distinct real roots, λ_1, λ_2 exist,
2. One (repeated) real root λ exists,
3. Complex conjugate roots $\alpha \pm i\beta$ exist;

and these correspond to the three canonical forms in Table 1, respectively. Multiples of the identity correspond to the first canonical form with $\lambda_1 = \lambda_2$.

Let $u, v \in \mathbb{R}^2$ be as indicated in Table 1. Let T be the matrix $(u \mid v)$, i.e. the matrix whose columns are u and v. Then T is the matrix that transforms the canonical form into A:

$$A = T\Lambda T^{-1},$$
$$e^{At} = Te^{\Lambda t}T^{-1}.$$

1.4 Chapter 5

A degenerate node is illustrated in Figure 1a. Here is an example:

Example 1.2. *Consider the ODE $\dot{y} = Ay$ where*

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Its characteristic equation is $\mu^2 - 2\mu + 1 = 0$, with (repeated) root $\mu = 1$. Since the matrix is not a multiple of the identity, there is only a single eigenvector (modulo scaling), u, corresponding to this eigenvalue 1. We deduce that the origin is an unstable degenerate node.

The eigenvector u satisfies

$$\begin{pmatrix} 2-1 & -1 \\ 1 & 0-1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We deduce that any non-zero vector parallel to $(1, 1)$ is an eigenvector of A; we choose $u = (1, 1)$.

The vector v satisfies $Av = 1v + u$, that is,

$$\begin{pmatrix} 2-1 & -1 \\ 1 & 0-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In other words, we require $v_1 - v_2 = 1$; we choose $v = (0, 1)$.

We deduce that the solution of the ODE is

$$y(t) = e^t \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} y(0).$$

Figure 5.2 on page 50 has the wrong orientation: the spiral should be anticlockwise as illustrated in Figure 1b.

The terms 'source'[2] and 'sink'[3] are used in the text but the complete definition is not given in one place:

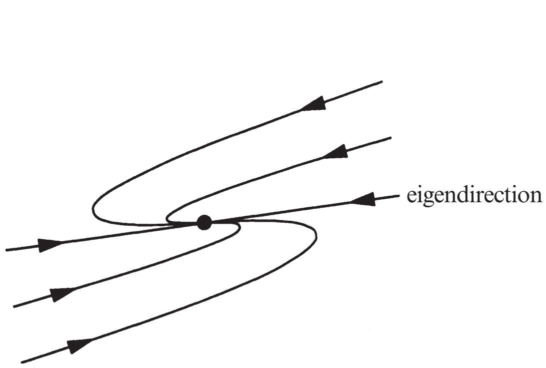
Definition 1.3 (Sources and sinks)**.**

1. *A source is an unstable equilibrium, i.e. either an unstable node, an unstable degenerate node or an unstable spiral.*
2. *A sink is an asymptotically stable equilibrium, i.e. either a stable node, a stable degenerate node or a stable spiral.*

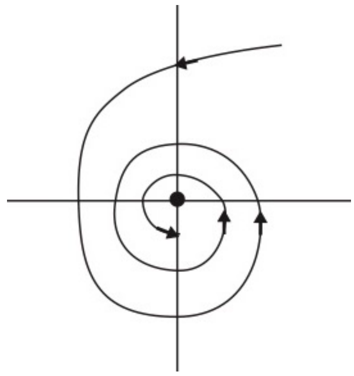
[2] This term is first used on page 48
[3] This term is first used on page 46

Eigenvalues of A (real unless stated otherwise)	Canonical Form Λ	$e^{\Lambda t}$	Asymptotically stable when	Stable but not asymptotically stable when	Unstable when	$\exists$ a basis $u, v \in \mathbb{R}^2$ such that	Name
λ_1, λ_2 (distinct)	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$	$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 \leqslant 0$ and $\lambda_1 \lambda_2 = 0$	$\lambda_1 > 0$ or $\lambda_2 > 0$	$Au = \lambda_1 u$ $Av = \lambda_2 v$	*Stable node* if $\lambda_1, \lambda_2 < 0$ *Unstable node* if $\lambda_1, \lambda_2 > 0$ *Saddle* if $\lambda_1 \lambda_2 < 0$
λ (repeated)	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$	$\lambda < 0$ (Fig. 1a)	$\lambda = 0$	$\lambda > 0$	$Au = \lambda u$ $Av = u + \lambda v$	*Stable/Unstable* *degenerate node*
$\alpha \pm i\beta$ ($\beta \neq 0$)	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	$e^{\alpha t} \begin{pmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{pmatrix}$	$\alpha < 0$ (Fig. 1b)	$\alpha = 0$	$\alpha > 0$	$Au = \alpha u + \beta v$ $Av = -\beta u + \alpha v$ Equivalently: $A(v + iu) = (\alpha + i\beta)(v + iu)$	*Stable spiral* if $\alpha < 0$ *Centre* if $\alpha = 0$ *Unstable spiral* if $\alpha > 0$

Table 1: Canonical forms and stability of the origin for invertible $A \in \mathbb{R}^{2\times 2}$ when A is not a multiple of the identity. Note that the table on page 42 lists e^{Λ} rather than $e^{\Lambda t}$.



(a) A stable ($\lambda < 0$) degenerate node.



(b) A corrected version of Figure 5.2 on page 50.

Figure 1: Two examples of asymptotically stable equilibria.

1.5 Chapter 6

Example 13 (pages 64-65). Equation (6.19) is the solution of the ODE in Equation (6.16) which passes through the origin. More generally, the solution passing through (x_0, y_0) is given by

$$xy - \frac{x^3}{3} = x_0 y_0 - \frac{x_0^3}{3}.$$

It is easy to verify that the flow presented in Equation (6.20) satisfies this condition.

1.6 Chapter 7

LaSalle Invariance Principle (page 75). It is important to note that $\mathcal{M}$ (defined after Equation (7.10)) is *not* the same as M defined in Equation (7.14).

Equation (7.19) on page 76 should read

$$E = \{(x, y) \in \mathcal{M} \mid y = 0\},$$

i.e. the "$\cap \mathcal{M}$" should be deleted.

1.7 Chapter 9

Equations (9.14) and (9.15) on page 95 should read:

$$\dot{r}^+ = \mu r^+ + a\left(r^+\right)^3 = 0,$$
$$\dot{\theta} = \omega + b\left(-\frac{\mu}{a}\right),$$

(note that there is no 't' in the second equation) and

$$\theta(t) = \left(\omega - \frac{\mu b}{a}\right) t + \theta(0)$$

(note that there is a 't' in the equation).

2 Solutions to selected problems

2.1 Chapter 2

Question 2.5. The ODE can be written as

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} t} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}. \tag{2.1}$$

Since this can be viewed as a map $\phi_t \colon \mathbb{R}^2 \to \mathbb{R}^2$ which maps $(x(0), y(0))$ to $(x(t), y(t))$, this is also the flow.

The matrix of sines and cosines in (2.1) above is a rotation matrix in $\mathbb{R}^2$, for an anti-clockwise rotation through an angle ωt. Since addition of angles corresponds to multiplication of rotation matrices, we have

$$\begin{aligned} \phi_{t+s}(x(0), y(0)) &= \begin{pmatrix} \cos(\omega(t+s)) & -\sin(\omega(t+s)) \\ \sin(\omega(t+s)) & \cos(\omega(t+s)) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} \cos(\omega s) & -\sin(\omega s) \\ \sin(\omega s) & \cos(\omega s) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \phi_t \circ \phi_s(x(0), y(0)). \end{aligned}$$

Thus the flow satisfies the time-shift property.

The initial condition for the time-shifted flow is

$$\phi_s(x(0), y(0)) = \begin{pmatrix} \cos(\omega s) & -\sin(\omega s) \\ \sin(\omega s) & \cos(\omega s) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Question 2.6. The ODE can be written as

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix above has eigenvalues λ and $-\lambda$:

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this we obtain,

$$\begin{aligned} \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}, \\ e^{\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} t} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix}. \end{aligned}$$

Thus the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Since this can be viewed as a map $\phi_t \colon \mathbb{R}^2 \to \mathbb{R}^2$ which maps $(x(0), y(0))$ to $(x(t), y(t))$, this is also the flow.

A calculation shows that

$$\begin{aligned} \phi_{t+s}(x(0), y(0)) &= \begin{pmatrix} \cosh(\lambda(t+s)) & \sinh(\lambda(t+s)) \\ \sinh(\lambda(t+s)) & \cosh(\lambda(t+s)) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix} \begin{pmatrix} \cosh(\lambda s) & \sinh(\lambda s) \\ \sinh(\lambda s) & \cosh(\lambda s) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \phi_t \circ \phi_s(x(0), y(0)). \end{aligned}$$

Thus the flow satisfies the time-shift property.

The initial condition for the time-shifted flow is

$$\phi_s(x(0), y(0)) = \begin{pmatrix} \cosh(\lambda s) & \sinh(\lambda s) \\ \sinh(\lambda s) & \cosh(\lambda s) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Question 2.7. Suppose, on the contrary, that trajectories beginning at p_1 and p_2 cross at q after time t:

$$\phi_t(p_1) = q = \phi_t(p_2).$$

Then, we obtain

$$p_1 = \phi_{-t}(q) = p_2,$$

so, in fact they are the same trajectory.

Questions 2.8 and 2.9. Let A and B be invariant sets, i.e.

$$p \in A \implies \phi_t(p) \in A, \quad \forall t \in \mathbb{R},$$

where ϕ_t is the flow; and similarly for B.

Suppose $p \in A \cup B$. Then either $p \in A$ *or* $p \in B$. Since A and B are invariant it follows that either $\phi_t(p) \in A \ \forall t \in \mathbb{R}$ *or* $\phi_t(p) \in B \ \forall t \in \mathbb{R}$. In other words, that $\phi_t(p) \in A \cup B \ \forall t \in \mathbb{R}$. Thus the union of two invariant sets is invariant.

Replacing the '*or*'s in the preceeding paragraph by 'and' we obtain a similar statement for the intersection of two invariant sets.

Question 2.10. Let A be a positive invariant set, i.e.

$$p \in A \implies \phi_t(p) \in A, \quad \forall t > 0,$$

where ϕ_t is the flow.

Let $q \in A^c$ and let $p = \phi_t(q)$ for some $t < 0$. Were $p \in A$, then $q = \phi_{-t}(p) \in A$ since A is positive invariant and $-t > 0$; this is a contradiction since $q \in A^c$. Thus $\phi_t(q) \in A^c \ \forall t < 0$, i.e. A^c is negative invariant.

2.2 Chapter 4

Question 4.3. Consider the function $x(t) = e^{At}x_0$. This function (i) exists for all time, (ii) satisfies the ODE (including initial condition), (iii) is infinitely differentiable with respect to the initial condition (although all derivatives higher than the first vanish).

2.3 Chapter 5

Question 5.3. (a) We calculate

$$\begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3i \end{pmatrix} = \begin{pmatrix} -1-3i \\ 9-3i \end{pmatrix} = (-1-3i) \begin{pmatrix} 1 \\ 3i \end{pmatrix},$$
$$\begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3i \end{pmatrix} = \begin{pmatrix} -1+3i \\ 9+3i \end{pmatrix} = (-1+3i) \begin{pmatrix} 1 \\ -3i \end{pmatrix}.$$

(b) We calculate

$$T_1^{-1}AT_1 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix},$$
$$T_2^{-1}AT_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix},$$
$$T_3^{-1}AT_3 = \begin{pmatrix} 0 & -\frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix},$$
$$T_4^{-1}AT_4 = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}.$$

(c) Let $u \pm iv$ be the eigenvectors of A. Then T above is one of $(u \mid -v)$, $(u \mid v)$, $(-v \mid u)$ and $(v \mid u)$.

2.4 Chapter 6

Question 6.1. (a) Let $p \in W^s(\bar{x})$. Then there exists $q \in W^s_{\text{loc}}(\bar{x})$ and $T < 0$ such that $p = \phi_T(q)$. Thus $\phi_t(p) = \phi_{t+T}(q) \in W^s(\bar{x})$. Thus $W^s(\bar{x})$ is invariant. A similar argument shows that $W^s(\bar{x})$ is invariant.

(b) Let $p \in W^s(\bar{x})$. Then there exists $q \in W^s_{\text{loc}}(\bar{x})$ and $T > 0$ such that $q = \phi_T(p)$. By definition of $W^s_{\text{loc}}(\bar{x})$, $\phi_t(q) \to \bar{x}$ as $t \to \infty$. Since $\phi_t(p) = \phi_{t-T}(q)$, it follows that $\phi_t(p) \to \bar{x}$ as $t \to \infty$. A similar argument holds for (c).

Question 6.2. The stable and unstable manifolds cannot intersect at an *isolated* point as illustrated in the figure since that would violate the uniqueness of the solution at the point.

However the global stable manifold can coincide with the global unstable manifold – this occurs for a homoclinic orbit.

2.5 Chapter 7

Question 7.1. Our first guess might be to consider

$$V(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \frac{1}{2}y^2 + \frac{1}{4}(x^2 - 1)^2 - \frac{1}{4}.$$

This vanishes at $(\pm 1, 0)$ but is not positive in neighbourhoods of $(\pm 1, 0)$ so we modify it to

$$V(x, y) = \frac{1}{2}y^2 + \frac{1}{4}(x^2 - 1)^2,$$

which satisfies the requirements of a Lyaponov function.

We compute,

$$\dot{V} = y\dot{y} + (x^2 - 1)x\dot{x} = y(x - x^3 - \delta y) + (x^2 - 1)xy = -\delta y^2 \leqslant 0.$$

It follows that $(\pm 1, 0)$ are stable when $\delta \geqslant 0$.

To show asymptotic stability when $\delta > 0$ we note that, when $\delta > 0$, $\dot{V} = 0$ only when $y = 0$. However when $y = 0$, $\dot{y} \neq 0$ (except at the equilibria) so $\dot{V}$ changes to become negative.

Question 7.2. See Question 6 below which considers the same system for $\epsilon = \frac{1}{2}$.

Question 7.5. The divergence of the vector field is $a + d$. Thus, by Bendixon's criterion, there are no periodic orbits if $a + d \neq 0$.

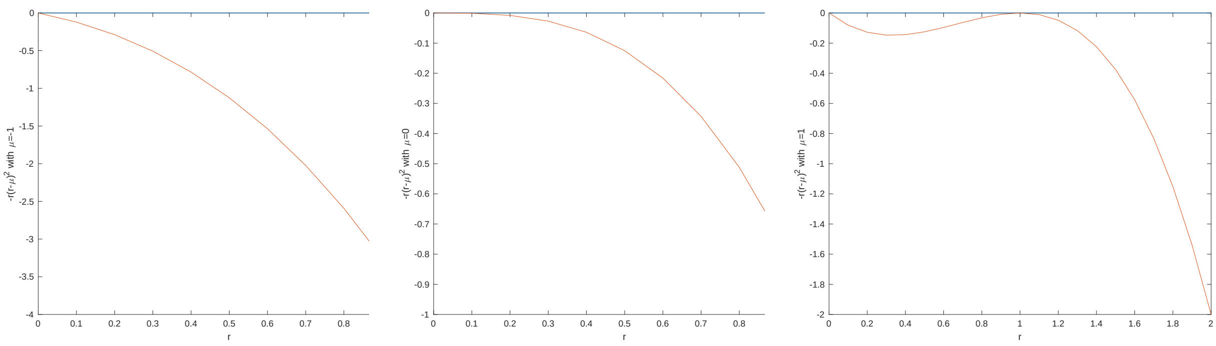
For all orbits to be periodic we need the eigenvalues of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to be purely imaginary. This requires that the sum of the eigenvalues, $a + d$, be 0 and the product of eigenvalues $ad - bc$, be positive.

We calculate,

$$\begin{aligned}
\dot{V} &= x\dot{x} + y\dot{y} \\
&= x(ax + by) + y(cx + dy) \\
&= ax^2 + (b + c)xy + dy^2 \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

Now, $(0, 0)$ is asymptotically stable if $\dot{V} < 0$ when $(x, y) \neq (0, 0)$. For this we need the matrix above to be negative-definite, i.e., for both its eigenvalues to be negative. This requires that its trace $a + d$ be negative and its determinant $ad - \left(\frac{b+c}{2}\right)^2$ be positive.

For the y-axis to be a stable manifold we need $(0, 1)$ to be an eigenvector—thus $b = 0$—with negative eigenvalue—thus $d < 0$. Similarly, for the x-axis to be an unstable manifold we need $(1, 0)$ to be an eigenvector—thus $c = 0$—with positive eigenvalue—thus $a > 0$.



(a) $\mu < 0$. (b) $\mu = 0$. (c) $\mu > 0$.

Figure 2: A plot of $-r(r-\mu)^2$ for Question 9.2.a.

Question 7.6. The matrix corresponding to the linearised system is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which corresponds to a centre. It follows that the linearised system has $\mathbb{R}^2$ as its centre manifold.

We calculate

$$\dot{V} = x\dot{x} + y\dot{y} = xy + y(-x - \frac{1}{2}x^2y) = -\frac{1}{2}x^2y^2 \leqslant 0, \tag{2.2}$$

from which it follows that $(0,0)$ is stable. This is consistent with the linearised analysis which could not determine the stability of the nonlinear system.

Let $c > 0$ and $\mathcal{M} := \{(x,y) \,|V(x,y) \leqslant c\}$, which is a closed and bounded set; since, from (2.2), $\dot{V} \leqslant 0$ it follows that it is also positively invariant.

Next, from (2.2), we calculate $E := \{(x,y) \,|\dot{V}(x,y) = 0\} = \{(x,y) \,|x = 0 \text{ or } y = 0\}$. The only point in E that remains in E is $(0,0)$, so it follows from the LeSalle Invariance Principle that every trajectory in $\mathcal{M}$ approaches $(0,0)$ as $t \to \infty$. Since c can be arbitrarily large this statement is true for arbitrary initial conditions. We conclude that $(0,0)$ is asymptotically stable.

2.6 Chapter 9

Note: In Figures 2-6 below the negative value of μ is chosen to be -1 and the positive value to be 1.

Question 9.2.a. From Figure 2, we conclude:

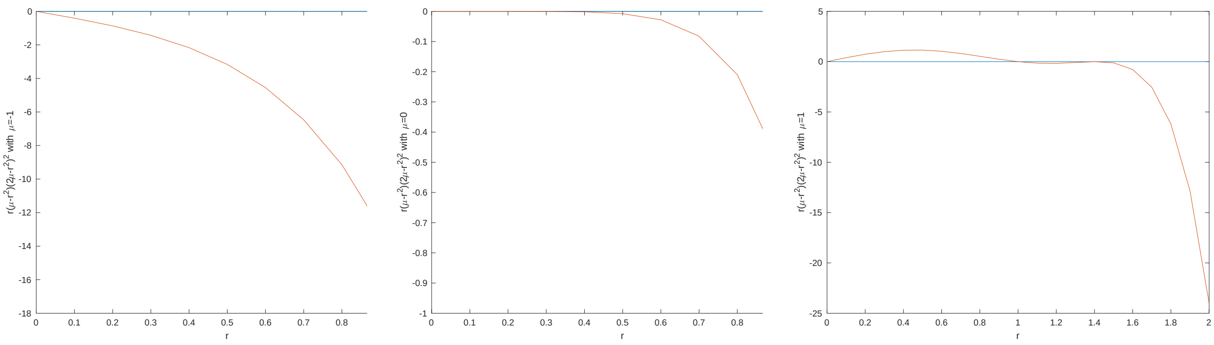
1. The origin is asymptotically stable for all $\mu \in \mathbb{R}$.
2. When $\mu > 0$, there exists an unstable periodic circular orbit with radius μ.

Question 9.2.b. From Figure 3, we conclude:

1. The origin is asymptotically stable when $\mu \leqslant 0$ and unstable when $\mu > 0$.
2. When $\mu > 0$, there exists an asymptotically stable periodic circular orbit with radius $\sqrt{\mu}$ and an unstable periodic circular orbit with radius $\sqrt{2\mu}$.

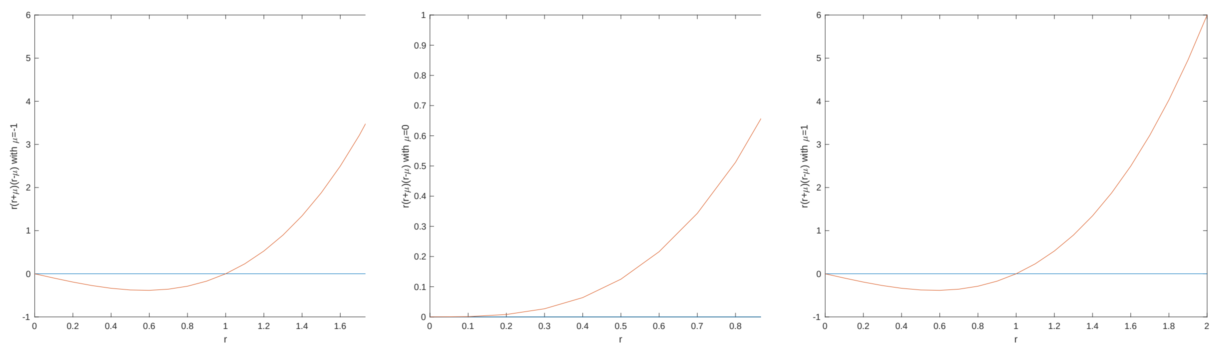
Question 9.2.c. From Figure 4, we conclude:

1. The origin is asymptotically stable when $\mu \neq 0$ and unstable when $\mu = 0$.
2. When $\mu \neq 0$, there exists an unstable periodic circular orbit with radius $|\mu|$.



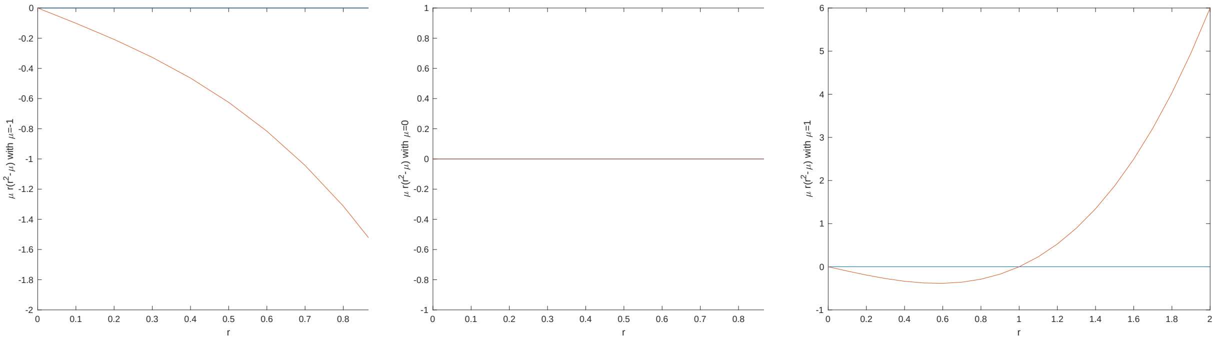
(a) $\mu < 0$. (b) $\mu = 0$. (c) $\mu > 0$.

Figure 3: A plot of $r(\mu - r^2)(2\mu - r^2)^2$ for Question 9.2.b.



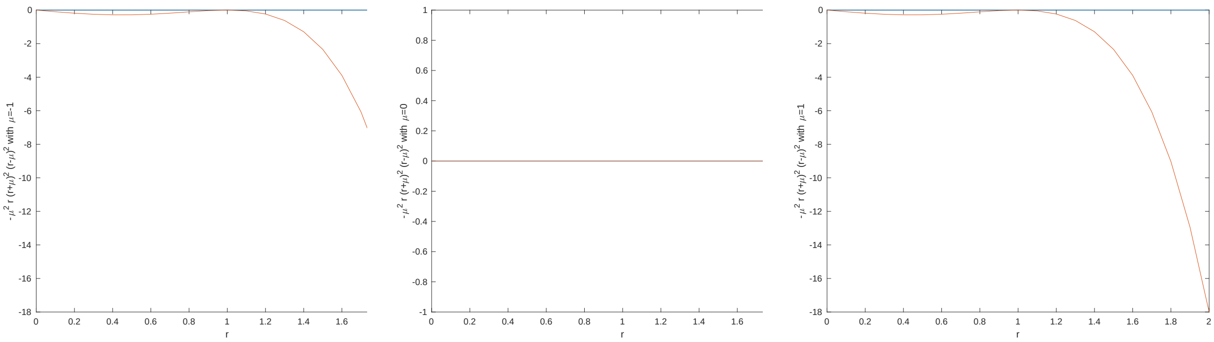
(a) $\mu < 0$. (b) $\mu = 0$. (c) $\mu > 0$.

Figure 4: A plot of $r(r + \mu)(r - \mu)$ for Question 9.2.c.



(a) $\mu < 0$. (b) $\mu = 0$. (c) $\mu > 0$.

Figure 5: A plot of $\mu r(r^2 - \mu)$ for Question 9.2.d.



(a) $\mu < 0$. (b) $\mu = 0$. (c) $\mu > 0$.

Figure 6: A plot of $-\mu^2 r(r + \mu)^2(r - \mu)^2$ for Question 9.2.e.

Question 9.2.d. From Figure 5, we conclude:

1. The origin is asymptotically stable when $\mu \neq 0$ and stable, but not asymptotically stable, when $\mu = 0$.

2. When $\mu > 0$, there exists an unstable periodic circular orbit with radius $\sqrt{\mu}$.

Question 9.2.e. From Figure 6, we conclude:

1. The origin is asymptotically stable when $\mu \neq 0$ and stable, but not asymptotically stable, when $\mu = 0$.

2. When $\mu \neq 0$, there exists an unstable periodic circular orbit with radius $|\mu|$.