UNIVERSITY OF BRISTOL

School of Mathematics

Ordinary Differential Equations 2
MATH20101
(Paper code MATH–20101J)

January 2019 2 hours 30 minutes

This paper contains **four** questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

 Do not turn over until instructed.

1. (**25 marks**)

   This question consists of two independent parts.

   (a) (**12 marks**)
   Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

   $$\dot{x} = \lambda y,$$
   $$\dot{y} = \lambda x,$$

   where $\lambda > 0$.

   i. (**8 marks**)
   Show that the resulting flow is given by

   $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

   Reminder: $\cosh(\lambda t) = \frac{1}{2}\left(e^{\lambda t} + e^{-\lambda t}\right)$ *and* $\sinh(\lambda t) = \frac{1}{2}\left(e^{\lambda t} - e^{-\lambda t}\right)$.

   ii. (**4 marks**)
   Compute the global stable and unstable manifolds of the equilibrium.

   (b) (**13 marks**)
   Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

   $$\dot{x} = -y^2,$$
   $$\dot{y} = -y.$$

   Compute the stable and centre manifolds of the origin.

2. (**25 marks**)

Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = y,$$
$$\dot{y} = x - x^3.$$

(a) (**4 marks**)
Compute the stability of the equilibrium at $(0, 0)$.

(b) (**2 marks**)
Show that the function

$$V(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

is constant along trajectories.

(c) (**7 marks**)
Sketch the phase portrait.

(d) (**6 marks**)
Are there any homoclinic orbits? If yes, how many? Indicate them on the phase portrait. What value(s) does V take on them?

(e) (**3 marks**)
Are there any periodic orbits, other than the equilibrium solutions?

(f) (**3 marks**)
Are there any heteroclinic orbits?

3. (**25 marks**)

Although this question is independent of Question 2, it is recommended that Question 2 be attempted before this question.

Consider the following ODE for $(x, y) \in \mathbb{R}^2$:

$$\dot{x} = y,$$
$$\dot{y} = x - x^3 - \alpha y,$$

where $\alpha = 1$.

(a) (**2 mark**)
Identify the equilibria.

(b) (**4 marks**)
Compute the stability of the equilibria other than $(0, 0)$.

(c) (**2 marks**)
Are there any non-equilibrium periodic orbits?

(d) (**15 marks**)
Use the LaSalle Invariance Principle to show that the trajectory starting at any $(x_0, y_0) \in \mathbb{R}^2$ approaches one of the equilibria as $t \to \infty$.
Hint: Use $V(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ *as a Lyapunov function.*

(e) (**2 marks**)
For which values of $\alpha \in \mathbb{R}$ do all answers to questions (3a) to (3d) above remain unchanged?

4. (**25 marks**)

Consider the ODE on $\mathbb{R}^2$ which in *polar coordinates* is given by

$$\dot{r} = r(\mu - r^2),$$
$$\dot{\theta} = 2\sin^2\frac{\theta}{2},$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) (**7 marks**)
Describe the bifurcations of the equilibria as μ is varied. What would be an appropriate name for this bifurcation?

(b) (**18 marks**)
Let $\mu = 1$ and $\phi_t(\cdot), t \in \mathbb{R}$ be the resulting flow.

i. (**6 marks**)
Without computing the divergence of the vector field, show that there exists a homoclinic orbit but no non-equilibrium periodic orbits.

ii. (**6 marks**)
Let (r_o, θ_o) be a point other than the origin. Show that

$$\lim_{t\to\infty} \phi_t(r_o, \theta_o) = (1, 0). \tag{1}$$

iii. (**6 marks**)
Is $(1, 0)$ a stable equilibrium? Explain why your answer is consistent with Equation (1) above.