UNIVERSITY OF BRISTOL

School of Mathematics

ORDINARY DIFFERENTIAL EQUATIONS 2

MATH 20101J (Paper code MATH–20101J)

January 2018   2 hours 30 minutes

This paper contains four questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= -x^2, \qquad (x, y) \in \mathbb{R}^2. \end{aligned}$$

   (a) (**5 marks**) Compute the flow generated by this vector field.

   (b) (**5 marks**) Can this vector field have periodic orbits? (You must justify your answer.)

   (c) (**5 marks**) Compute the centre manifold of the origin.

   (d) (**5 marks)** Compute the stable manifold of the origin.

   (e) (**5 marks)** Show that the origin is Lyapunov stable.

2. Consider the following autonomous vector field on $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - x^2y + zxy, \\ \dot{z} &= -z + xz^2, \qquad (x, y, z) \in \mathbb{R}^3. \end{aligned}$$

(a) (**5 marks**) Show that $(x, y, z) = (0, 0, 0)$ is a nonhyperbolic fixed point.

(b) (**5 marks**) Compute the stable manifold of $(x, y, z) = (0, 0, 0)$.

(c) (**5 marks**) Compute the centre manifold of $(x, y, z) = (0, 0, 0)$.

(d) (**10 marks**) Use the centre manifold reduction and the LaSalle invariance principle restricted to the centre manifold to show that $(x, y, z) = (0, 0, 0)$ is asymptotically stable.

3. Consider the following autonomous vector field on the plane:

$$\begin{aligned}
\dot{x} &= \mu x - \omega y - \frac{x}{2}\left(x^2 + y^2 - \frac{2}{3}(x^2+y^2)^{\frac{3}{2}}\right), \\
\dot{y} &= \omega x + \mu y - \frac{y}{2}\left(x^2 + y^2 - \frac{2}{3}(x^2+y^2)^{\frac{3}{2}}\right),
\end{aligned}$$

where $(x, y) \in \mathbb{R}^2$, $\omega > 0$ and $\mu \in \mathbb{R}$ is a variable parameter.

(a) (**5 marks**) Show that $(x, y) = (0, 0)$ is a nonhyperbolic fixed point for $\mu = 0$.

(b) (**5 marks**) Transform the vector field to polar coordinates using $x = r\cos\theta$, $y = r\sin\theta$.

(c) (**5 marks**) Determine the nature of the change of stability and bifurcation at $(x, y) = (0, 0)$.

(d) (**10 marks**) Determine all periodic orbits, their stability, and bifurcations.

4. Consider the following autonomous vector field:

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= \sin y, \end{aligned}$$

where $x \in \mathbb{R}$, $-\pi \leq y \leq \pi$.

(a) (**5 marks**) Find all fixed points.

(b) (**5 marks**) Determine the linearized stability properties of each fixed point.

(c) (**5 marks**) Sketch the phase portrait.

(d) (**5 marks**) Determine the global stable and unstable manifolds of the origin.

(e) (**5 marks**) Determine whether any nonhyperbolic fixed points are stable or unstable? (You must justify your answer.)

End of examination.