UNIVERSITY OF BRISTOL

School of Mathematics

ORDINARY DIFFERENTIAL EQUATIONS 2

MATH 20101J (Paper code MATH–20101J)

January 2017 2 hours 30 minutes

This paper contains four questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= ax + by, \\ \dot{y} &= cx + dy, \qquad (x, y) \in \mathbb{R}^2, \end{aligned} \tag{1}$$

where a, b, c, $d \in \mathbb{R}$. In the questions below you are asked to give conditions on the constants a, b, c, and d so that particular dynamical phenomena are satisfied. You do *not* have to give all possible conditions on the constants in order for the dynamical condition to be satisfied. One condition will be sufficient, but you must justify your answer.

(a) (**5 marks**) Give conditions on a, b, c, d for which the vector field has no periodic orbits.

(b) (**5 marks**) Give conditions on a, b, c, d for which all of the orbits are periodic.

(c) (**5 marks**) Using

$$V(x, y) = \frac{1}{2}(x^2 + y^2)$$

as a Lyapunov function, give conditions on a, b, c, d for which $(x, y) = (0, 0)$ is asymptotically stable.

(d) (**10 marks)** Give conditions on a, b, c, d for which $x = 0$ is the stable manifold of $(x, y) = (0, 0)$ and $y = 0$ is the unstable manifold of $(x, y) = (0, 0)$.

2. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -y + x^2, \qquad (x, y) \in \mathbb{R}^2 \end{aligned} \tag{2}$$

(a) (**10 marks**) Compute the flow generated by (2).

(b) (**5 marks**) Compute the stable and unstable subspaces for the linearization of (2) about $(x, y) = (0, 0)$ and show that they are invariant under the linearized dynamics.

(c) (**5 marks**) Compute the stable manifold of the origin and sketch it.

(d) (**5 marks**) Compute the unstable manifold of the origin and sketch it.

3. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \frac{x^2 y}{2}, \qquad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{3}$$

(a) (**5 marks**) Determine the linearized stability of $(x, y) = (0, 0)$.

(b) (**5 marks**) Describe the invariant manifold structure for the linearization of (3) about $(x, y) = (0, 0)$.

(c) (**5 marks**) Using $V(x, y) = \frac{1}{2}(x^2 + y^2)$ as a Lyapunov function, what can you conclude about the stability of the origin? Does this agree with part (a)? Why or why not?

(d) (**10 marks**) Using the LaSalle invariance principle, determine the fate of a trajectory starting at an arbitrary initial condition as $t \to \infty$? What does this result allow you to conclude about stability of $(x, y) = (0, 0)$?

4. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \alpha xy, \qquad (x, y) \in \mathbb{R}^2, \end{aligned} \tag{4}$$

where $\alpha \in \mathbb{R}$ is a real parameter.

(a) (**6 marks**) Determine the equilibria of (4) and their linearized stability.

(b) (**4 marks**) Discuss the types of bifurcations of equilibria for (4) that can occur as α is varied.

Consider the following autonomous vector field on the $\mathbb{R}$:

$$\dot{x} = \mu - \frac{x^2}{2} + \frac{x^4}{4}, \qquad x \in \mathbb{R}, \tag{5}$$

where $\mu \in \mathbb{R}$ is a real parameter.

(c) (**3 marks**) Sketch the equilibria of (5) in the $\mu - x$ plane.

(d) (**6 marks**) Determine the types of bifurcations of equilibria that may occur, and their location in the $\mu - x$ plane.

(e) (**6 marks**) Sketch the complete bifurcation diagram for (5) in the $\mu - x$ plane, and indicate the stability properties of all equilibria.

End of examination.