UNIVERSITY OF BRISTOL

School of Mathematics

MATH 20101
ORDINARY DIFFERENTIAL EQUATIONS 2

January 2016

2 hours and 30 minutes

This paper contains **four** *questions.*
All answers will be used for assesment.

Calculators are **NOT** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed

1. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= 2x^2, \qquad (x, y) \in \mathbb{R}^2. \end{aligned} \tag{1}$$

   (a) (**5 marks**) Compute the flow generated by this vector field.

   (b) (**2 marks**) Show that (1) has no periodic orbits.

   (c) (**2 marks**) Determine all of the nonhyperbolic fixed points for (1).

   (d) (**4 marks**) Show that $x = 0$ and $y = x^2 + c, \quad c \in \mathbb{R}$, are invariant manifolds for (1).

   (e) (**8 marks**) Describe how the invariant manifold structure of (1) restricts the motion of trajectories in the phase plane

   (f) (**4 marks**) Sketch the phase portrait for (1).

2. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= xy^2, \\ \dot{y} &= -y + x^2, \qquad (x, y) \in \mathbb{R}^2 \end{aligned} \tag{2}$$

(a) (**5 marks**) Show that $(x, y) = (0, 0)$ is a nonhyperbolic fixed point.

(b) (**5 marks**) Compute the stable and center subspaces for the linearization of (2) about $(x, y) = (0, 0)$ and show that they are invariant under the linearized dynamics.

(c) (**5 marks**) Compute an approximation to the center manifold of the origin through second order terms.

(d) (**5 marks**) Sketch the flow on the center manifold near the origin.

(e) (**5 marks**) Use the center manifold to determine the nature of the stability of $(x, y) = (0, 0)$ for (2).

3. Consider the following autonomous vector field on the plane:

$$\begin{aligned} \dot{x} &= \mu x - y - x(x^2 + y^2)^3, \\ \dot{y} &= x + \mu y - y(x^2 + y^2)^3, \qquad (x, y) \in \mathbb{R}^2, \end{aligned} \tag{3}$$

where $\mu \in \mathbb{R}$ is a real parameter.

(a) (**4 marks**) Discuss the types of bifurcations of equilibria for (3) that can occur as μ is varied.

(b) (**6 marks**) Sketch the bifurcation diagram.

Consider the following autonomous vector field on the $\mathbb{R}$:

$$\dot{x} = (x - \mu)(x - \frac{\mu}{2}), \qquad x \in \mathbb{R}, \tag{4}$$

where $\mu \in \mathbb{R}$ is a real parameter.

(c) (**5 marks**) Sketch the equilibria of (4) in the $\mu - x$ plane.

(d) (**10 marks**) Sketch the complete bifurcation diagram for (4) in the $\mu - x$ plane, and indicate the stability properties of all equilibria.