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Chapter 6

3. $$\dot{x} = \alpha x$$
$$\dot{y} = \beta y + \gamma x^{n+1}$$

$\alpha < 0$, $\beta > 0$, γ is a real number and n is a positive integer.

a) Jacobian associated with the linearization about the origin.

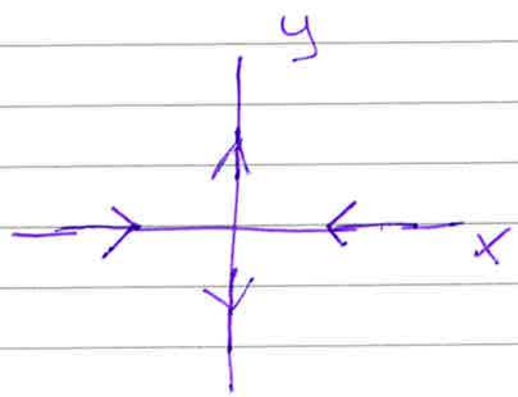
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

For $\alpha < 0$, $\beta > 0$ the origin is a hyperbolic, saddle.

b) Vector field linearized about the origin.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\alpha < 0$, $\beta > 0$



$E^{u.} = \{ (x,y) \mid x = 0 \}$

$E^{s.} = \{ (x,y) \mid y = 0 \}$

c) $x = 0 \Rightarrow \dot{x} = 0 \Rightarrow E^{u}$ invariant

$y = 0 \Rightarrow \dot{y} = 0 \Rightarrow E^{s.}$ invariant

d) $X(t, X_0) = X_0 e^{\alpha t}$

$$\dot{y} = \beta y + \gamma X_0^{n+1} e^{\alpha(n+1)t}$$

Let $y = u e^{\beta t}$ $\quad (\Rightarrow y(0) = u(0))$

$$\dot{y} = \dot{u} e^{\beta t} + \cancel{\beta u e^{\beta t}} = \beta y + \gamma X_0^{n+1} e^{\alpha(n+1)t}$$
$$= \cancel{\beta u e^{\beta t}} + \gamma X_0^{n+1} e^{\alpha(n+1)t}$$

$$\dot{u} e^{\beta t} = \gamma X_0^{n+1} e^{\alpha(n+1)t}$$

$$\dot{u} = \gamma X_0^{n+1} e^{[\alpha(n+1)-\beta]t}$$

$$u(t) = u(0) + \int_0^t \gamma X_0^{n+1} e^{(\alpha(n+1)-\beta)t} dt$$

$$= u(0) + \gamma X_0^{n+1} \frac{1}{\alpha(n+1)-\beta} e^{(\alpha(n+1)-\beta)t} \Big|_0^t$$

$$y(t)\, e^{-\beta t} = y_0 + \frac{\gamma x_0^{n+1}}{\alpha(n+1)-\beta}\left[e^{(\alpha(n+1)-\beta)t} - 1 \right]$$

or

$$y(t) = \left(y_0 - \frac{\gamma x_0^{n+1}}{\alpha(n+1)-\beta} \right) e^{\beta t} + \frac{\gamma x_0^{n+1}}{\alpha(n+1)-\beta}\, e^{\alpha(n+1)t}$$

e) Global unstable manifold.

$x = 0$

Global stable manifold

choose initial conditions such that trajectories through those initial conditions go to zero as $t \to +\infty$.

$\Rightarrow$ the coefficient on the $e^{\beta t}$ term of the y component of the flow must vanish.

$$\Rightarrow \quad y = \frac{\gamma x^{n+1}}{\alpha(n+1)-\beta}$$

f) . $X=0$ is clearly invariant for the nonlinear vector field since $X=0 \Rightarrow \dot{X}=0$

Check that the vector field is tangent to

$$y = \frac{\gamma}{\alpha(n+1)-\beta} X^{n+1}$$

$$\dot{y} = \frac{\gamma}{\alpha(n+1)-\beta} (n+1) X^{n} \dot{X}$$

Substitute the vector field into this relation and check that equality holds.

$$\dot{y} = \beta y + \gamma X^{n+1}$$

$$= \frac{\gamma\beta}{\alpha(n+1)-\beta} X^{n+1} + \gamma X^{n+1}$$

$$= \left[\frac{\gamma\beta}{\alpha(n+1)-\beta} + \frac{\gamma(\alpha(n+1)-\beta)}{\alpha(n+1)-\beta} \right] X^{n+1}$$

$$= \alpha \frac{\gamma(n+1)}{\alpha(n+1)-\beta} X^{n+1} \quad (*)$$

Now

$$\frac{\gamma}{\alpha(n+1)-\beta} (n+1) X^{n} X^{0}$$

$$= \alpha \frac{\gamma(n+1)}{\alpha(n+1)-\beta} X^{n+1} \quad (**)$$

Now $(*) = (**)$

Therefore

$$y = \frac{\gamma}{\alpha(n+1)-\beta} X^{n+1}$$

is invariant.

g) The unstable manifold

$x = 0$

does not depend on γ or n.

The stable manifold is the graph of

$$y = \frac{\gamma}{\alpha(n+1) - \beta} x^{n+1}$$

Now $\alpha < 0$, $\beta > 0$ $\Rightarrow$ $\alpha(n+1) - \beta < 0$

$\gamma = 0$ is a trivial case. This implies that the global stable manifold is the x axis. (and that the vector field is linear).

Two Cases

$$\gamma > 0 \implies \frac{\gamma}{\alpha(n+1) - \beta} < 0$$

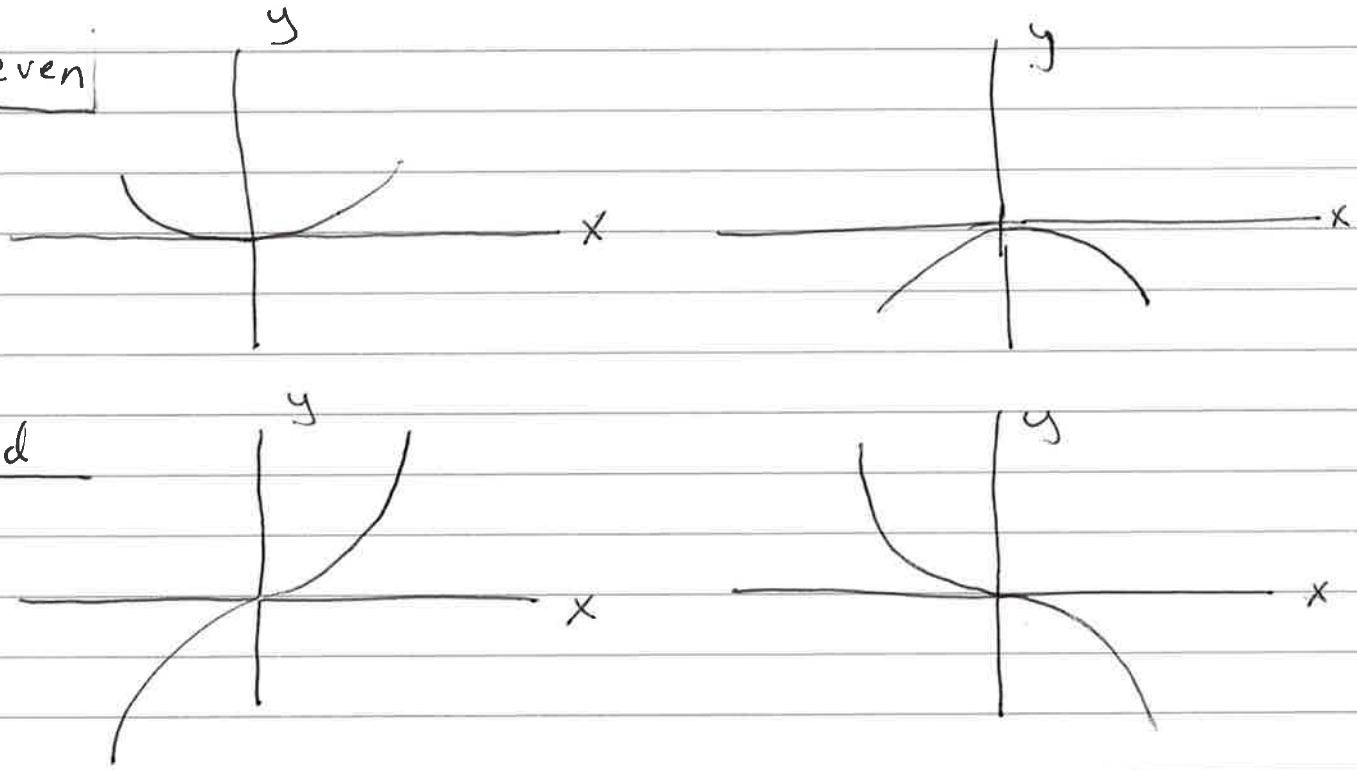
$$\gamma < 0 \implies \frac{\gamma}{\alpha(n+1) - \beta} > 0$$

n is a positive integer

$\implies$ $n+1$ is even or odd.

n+1 even

n+1 odd



$\boxed{\gamma < 0}$ $\boxed{\gamma > 0}$

4. Let $X(t)$ denote the homoclinic orbit. Then

$$\lim_{t \to \infty} X(t) = X_0$$

$$\lim_{t \to -\infty} X(t) = X_0$$

Or, alternatively, $X(t)$ is in the stable and unstable manifold of X_0

5. Let $X(t)$ denote the heteroclinic orbit. Then

$$\lim_{t \to \infty} X(t) = X_0$$

$$\lim_{t \to -\infty} X(t) = X_1$$

Or, alternatively, $X(t)$ is in the stable manifold of X_0 and the unstable manifold of X_1.