Chapter 5

Q1

$A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}$ is an eigenvalue w/ eigenvector $e \in \mathbb{C}^n$:

$$Ae = \lambda e$$

Conjugating both sides:

$$\overline{Ae} = \overline{\lambda e} = \bar{\lambda}\bar{e}$$

Now $\overline{Ae} = \bar{A}\,\bar{e} = A\bar{e}$ (since $A = \bar{A}$, seeing that A is real). Thus

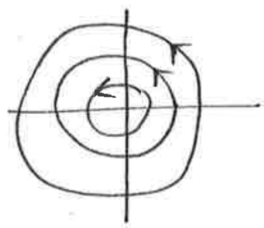
$$A\bar{e} = \bar{\lambda}\bar{e}$$

Thus $\bar{e} \in \mathbb{C}^n$ is an eigenvector of A with eigenvalue $\bar{\lambda}$.

Q2 (a) $A_1 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, $\omega > 0$

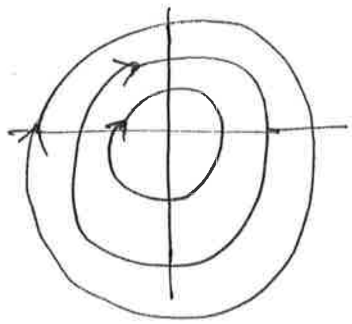
Comparing with the canonical form $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ we see that $\alpha = 0$ and $\beta = \omega > 0$.

Since $\alpha = 0$ the trajectories are circles; since $\beta > 0$ they are anticlockwise:



(b) $A_2 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$, $\omega > 0$

As in (a) above but now $\beta = -\omega < 0$ so the trajectories are clockwise:



4. $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues

$$\det \begin{pmatrix} -2-\lambda & 1 \\ -5 & 2-\lambda \end{pmatrix} = (-2-\lambda)(2-\lambda)+5=0$$

or, $\lambda^2+1=0 \Rightarrow \lambda = \pm i$

eigenvectors

$$\begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left.\begin{aligned} -2x_1 + x_2 &= -i x_1 \\ -5x_1 + 2x_2 &= -i x_2 \end{aligned}\right\} \Rightarrow \quad x_2 = (2-i)x_1$$

$$\begin{pmatrix} 1 \\ 2-i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$T^{-1} A T = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

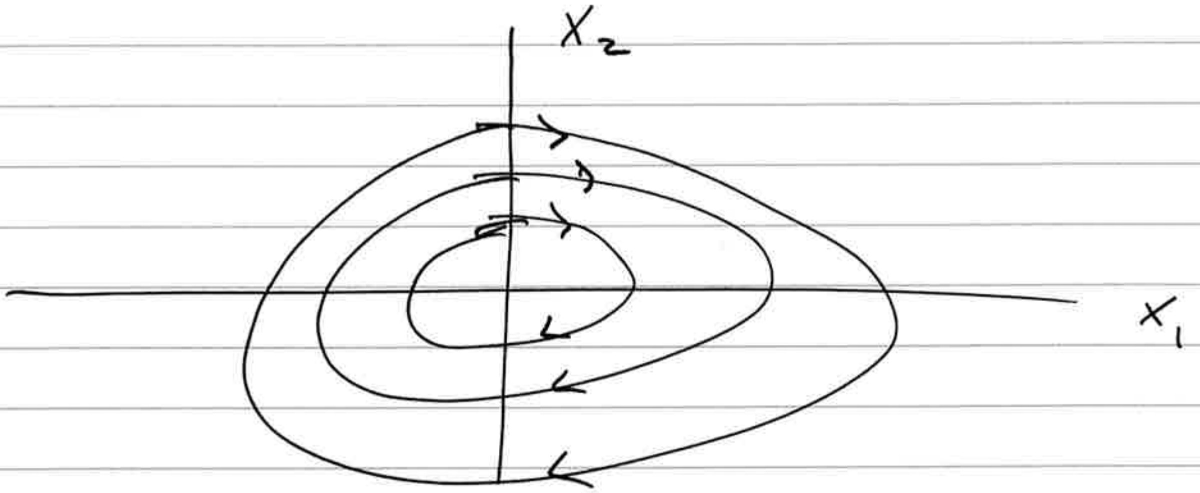
$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \Lambda$$

$$e^{\Lambda} = \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix}$$

$$e^{A} = T e^{\Lambda} T^{-1}$$

The origin is a <u>centre</u>, i.e. the eigenvalues of A are purely imaginary.



<u>Lyapunov stable</u>

(Why are the directions of the arrows as shown?)

5

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

<u>eigenvalues</u>

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 1 \pm \tfrac{1}{2}\sqrt{4+12} = 3, -1$$

<u>eigenvectors</u>

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_2 &= 3x_1 \\ 2x_1 + x_2 &= 3x_2 \end{aligned}\right\} \Rightarrow x_1 = x_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_2 &= -x_1 \\ 2x_1 + x_2 &= -x_2 \end{aligned}\right\} \Rightarrow x_2 = -x_1 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$T^{-1}AT = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}$$

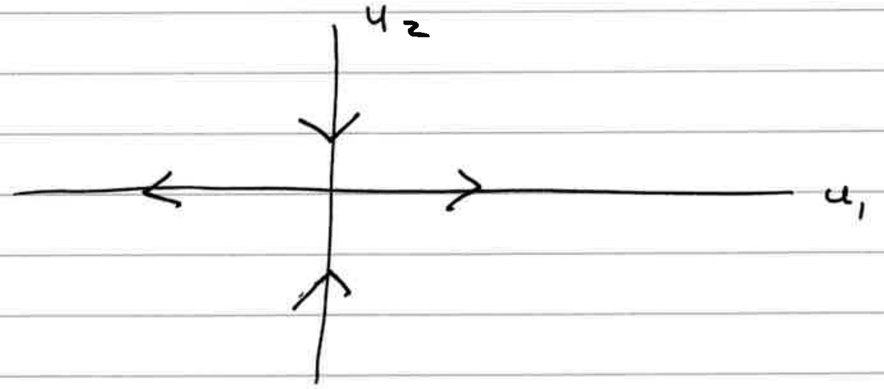
$$= -\frac{1}{2}\begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \checkmark$$

$$e^{At} = -\frac{1}{2}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & +1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} -e^{3t} & -e^{3t} \\ -e^{-t} & +e^{-t} \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -e^{3t} - e^{-t} & -e^{3t} + e^{-t} \\ -e^{3t} + e^{-t} & -e^{3t} - e^{-t} \end{pmatrix}$$

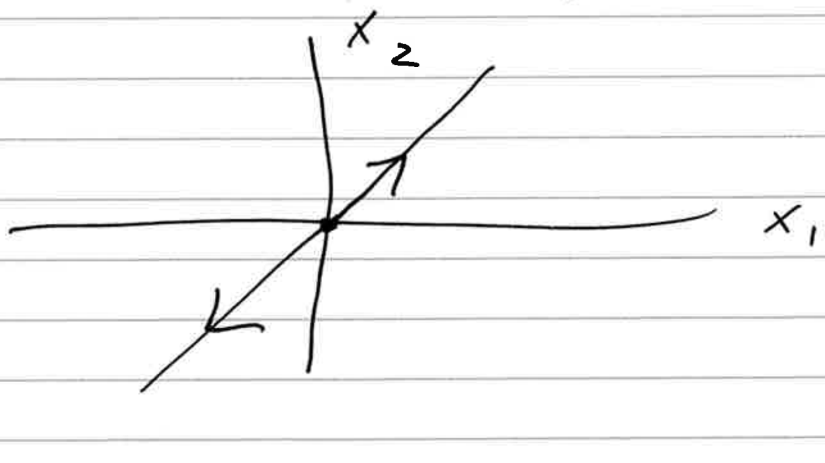
$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \quad T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$u_1 = x_1 , \quad u_1 = x_2 , \quad x_1 = x_2$$

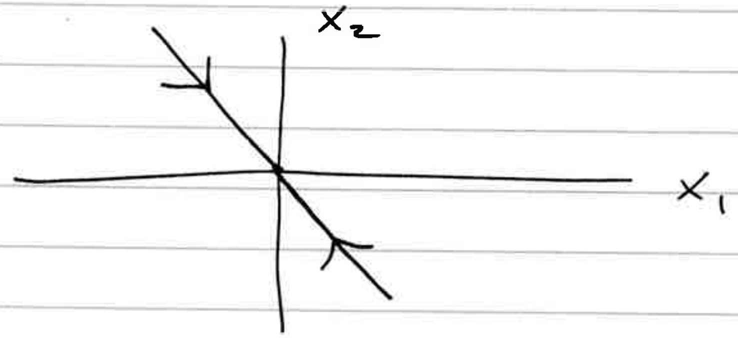


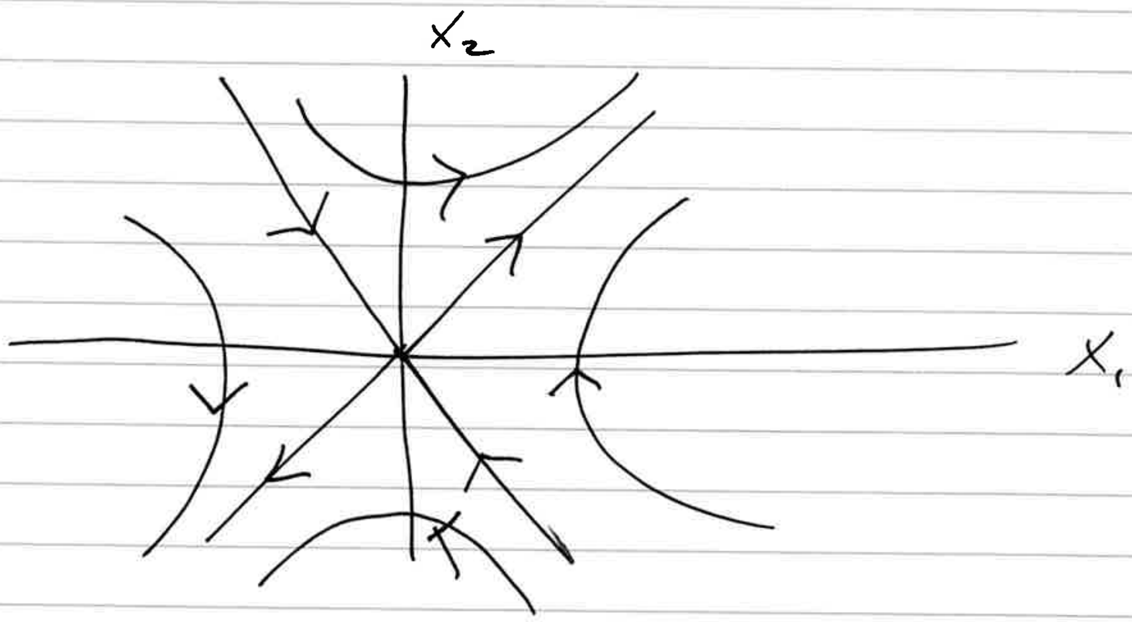
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x_1 = u_2$

$x_2 = -u_2$

$x_1 = -x_2$





The origin is unstable.

It is a <u>saddle</u>

6. $$\underbrace{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{S} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{N}$$

$A = S + N$, $SN = NS$

$$e^A = \mathbb{1} + (S+N) + \frac{1}{2!}(S+N)^2 + \frac{1}{3!}(S+N)^3 + \cdots + \frac{1}{n!}(S+N)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(S+N)^n$$

Using the binomial expansion

$$(S+N)^n = \sum_{k=0}^{n} \binom{n}{k} S^k N^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

or

$$(S+N)^n = \binom{n}{0} S^n N^0 + \binom{n}{1} S^{n-1} N + \binom{n}{2} S^{n-2} N^2 + \cdots + \binom{n}{n-1} S N^{n-1} + \binom{n}{n} S^0 N^n$$

Note that $N^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $n \geq 2$, $(N^0 = \mathbb{1})$

$$(S+N)^n = S^n + n S^{n-1} N \quad , \quad n \geq 1$$

$$e^A = \sum_{n=1}^{\infty} \frac{1}{n!} \left[S^n + n S^{n-1} N \right] + \mathbb{1}$$

or, $$e^{A} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{S^{n}}{n!}$$

$$+ N \sum_{n=1}^{\infty} \frac{S^{n-1}}{(n-1)!}$$

$$= e^{S} (\mathbb{1} + N)$$

$$= \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{pmatrix}$$