MATH20101 Ordinary Differential Equations 2

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These notes are intended as a supplement to the textbook for the unit (Stephen Wiggins, Ordinary Differential Equations).[1]

Contents

1 Notes

1.1 Chapter 2

The word 'orbit' is used for the first time on page 29 (Questions 1-4) but is not defined in the text:

Definition 1.1 (Orbit)**.** *Let p be a point on the phase space. The orbit of p is the set*

$$\{\phi_t(p) \mid t \in \mathbb{R}\}$$

where $\phi_t(\cdot)$ is the flow (see pages 24-25).

1.2 Chapter 3

In Definition 16 on page 33 the term 'neighbourhood' means "sufficiently small open set".

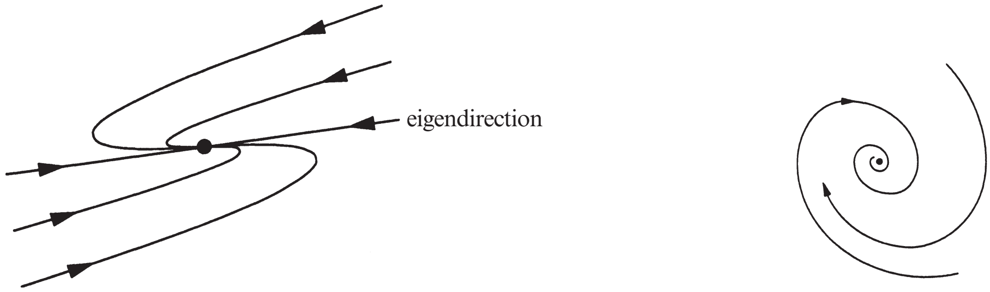
1.3 Chapters 4 and 5

Given $A \in \mathbb{R}^{2\times 2}$, *which is not a multiple of the identity*, we identify its canonical form by considering the roots of the characteristic polynomial

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0.$$

Three case arise:

1. Two distinct real roots, λ_1, λ_2 exist,



(a) A stable ($\lambda < 0$) degenerate node. (b) A spiral with $\alpha, \beta < 0$.

Figure 1: Two examples of asymptotically stable equilibria.

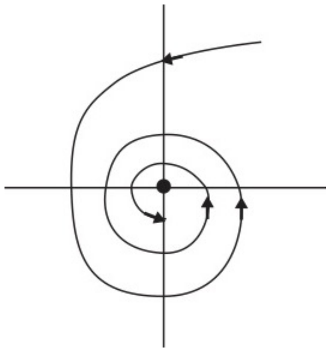


Figure 2: A corrected version of Figure 5.2 on page 50.

2. One (repeated) real root λ exists,

3. Complex conjugate roots $\alpha \pm i\beta$ exist;

and these correspond to the three canonical forms in Table 1, respectively. Multiples of the identity correspond to the first canonical form with $\lambda_1 = \lambda_2$.

Let $u, v \in \mathbb{R}^2$ be as indicated in Table 1. Let T be the matrix $(u \mid v)$, i.e. the matrix whose columns are u and v. Then T is the matrix that transforms the canonical form into A:

$$A = T\Lambda T^{-1},$$
$$e^{At} = Te^{\Lambda t}T^{-1}.$$

The terms 'source'[2] and 'sink'[3] are used in the text but the complete definition is not given in one place:

Definition 1.2 (Sources and sinks)**.**

1. *A source is an unstable equilibrium, i.e. either an unstable node, an unstable degenerate node or an unstable spiral.*

2. *A sink is an asymptotically stable equilibrium, i.e. either a stable node, a stable degenerate node or a stable spiral.*

1.4 Chapter 5

Figure 5.2 on page 50 has the wrong orientation: the spiral should be anticlockwise as illustrated in Figure 2.

[2] This term is first used on page 48
[3] This term is first used on page 46

Eigenvalues of A (real unless indicated otherwise)	Canonical Form Λ	$e^{\Lambda t}$	Asymptotically stable when	Stable but not asymptotically stable when	Unstable when	$\exists$ a basis $u, v \in \mathbb{R}^2$ such that	Name
λ_1, λ_2 (distinct)	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$	$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 \leqslant 0$ and $\lambda_1 \lambda_2 = 0$	$\lambda_1 > 0$ or $\lambda_2 > 0$	$Au = \lambda_1 u$ $Av = \lambda_2 v$	*Stable node* if $\lambda_1, \lambda_2 < 0$ *Unstable node* if $\lambda_1, \lambda_2 > 0$ *Saddle* if $\lambda_1 \lambda_2 < 0$
λ (repeated)	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$	$\lambda < 0$ (Fig. 1a)	$\lambda = 0$	$\lambda > 0$	$Au = \lambda u$ $Av = u + \lambda v$	*Stable/Unstable degenerate node*
$\alpha \pm i\beta$ ($\beta \neq 0$)	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	$e^{\alpha t} \begin{pmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{pmatrix}$	$\alpha < 0$ (Fig. 1b)	$\alpha = 0$	$\alpha > 0$	$Au = \alpha u + \beta v$ $Av = -\beta u + \alpha v$	*Stable spiral* if $\alpha < 0$ *Centre* if $\alpha = 0$ *Unstable spiral* if $\alpha > 0$

Table 1: Canonical forms and stability of the origin for $A \in \mathbb{R}^{2\times 2}$ when $A \neq I$. Note that the table on page 42 lists e^{Λ} rather than $e^{\Lambda t}$.

2 Solutions to selected problems

2.1 Chapter 2

Question 5. The ODE can be written as

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} t} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}. \tag{2.1}$$

Since this can be viewed as a map $\phi_t \colon \mathbb{R}^2 \to \mathbb{R}^2$ which maps $(x(0), y(0))$ to $(x(t), y(t))$, this is also the flow.

The matrix of sines and cosines in (2.1) above is a rotation matrix in $\mathbb{R}^2$, for an anti-clockwise rotation through an angle ωt. Since addition of angles corresponds to multiplication of rotation matrices, we have

$$\begin{aligned} \phi_{t+s}(x(0), y(0)) &= \begin{pmatrix} \cos(\omega(t+s)) & -\sin(\omega(t+s)) \\ \sin(\omega(t+s)) & \cos(\omega(t+s)) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} \cos(\omega s) & -\sin(\omega s) \\ \sin(\omega s) & \cos(\omega s) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \phi_t \circ \phi_s(x(0), y(0)). \end{aligned}$$

Thus the flow satisfies the time-shift property.

The initial condition for the time-shifted flow is

$$\phi_s(x(0), y(0)) = \begin{pmatrix} \cos(\omega s) & -\sin(\omega s) \\ \sin(\omega s) & \cos(\omega s) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Question 6. The ODE can be written as

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix above has eigenvalues λ and $-\lambda$:

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this we obtain,

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1},$$
$$e^{\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} t} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix}.$$

Thus the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) \\ \sinh(\lambda t) & \cosh(\lambda t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Since this can be viewed as a map $\phi_t \colon \mathbb{R}^2 \to \mathbb{R}^2$ which maps $(x(0), y(0))$ to $(x(t), y(t))$, this is also the flow.

A calculation shows that

$$\begin{aligned}\phi_{t+s}(x(0),y(0)) &= \begin{pmatrix}\cosh(\lambda(t+s)) & \sinh(\lambda(t+s))\\ \sinh(\lambda(t+s)) & \cosh(\lambda(t+s))\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}\\ &= \begin{pmatrix}\cosh(\lambda t) & \sinh(\lambda t)\\ \sinh(\lambda t) & \cosh(\lambda t)\end{pmatrix}\begin{pmatrix}\cosh(\lambda s) & \sinh(\lambda s)\\ \sinh(\lambda s) & \cosh(\lambda s)\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \phi_t \circ \phi_s(x(0),y(0)).\end{aligned}$$

Thus the flow satisfies the time-shift property.

The initial condition for the time-shifted flow is

$$\phi_s(x(0),y(0)) = \begin{pmatrix}\cosh(\lambda s) & \sinh(\lambda s)\\ \sinh(\lambda s) & \cosh(\lambda s)\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}.$$

Question 7. Suppose, on the contrary, that trajectories beginning at p_1 and p_2 cross at q after time t:

$$\phi_t(p_1) = q = \phi_t(p_2).$$

Then, we obtain

$$p_1 = \phi_{-t}(q) = p_2,$$

so, in fact they are the same trajectory.

Questions 8 and 9. Let A and B be invariant sets, i.e.

$$p \in A \implies \phi_t(p) \in A, \quad \forall t \in \mathbb{R},$$

where ϕ_t is the flow; and similarly for B.

Suppose $p \in A \cup B$. Then either $p \in A$ *or* $p \in B$. Since A and B are invariant it follows that either $\phi_t(p) \in A$ $\forall t \in \mathbb{R}$ *or* $\phi_t(p) \in B$ $\forall t \in \mathbb{R}$. In other words, that $\phi_t(p) \in A \cup B$ $\forall t \in \mathbb{R}$. Thus the union of two invariant sets is invariant.

Replacing the '*or*'s in the preceeding paragraph by 'and' we obtain a similar statement for the intersection of two invariant sets.

Question 10. Let A be a positive invariant set, i.e.

$$p \in A \implies \phi_t(p) \in A, \quad \forall t > 0,$$

where ϕ_t is the flow.

Let $q \in A^c$ and let $p = \phi_t(q)$ for some $t < 0$. Were $p \in A$, then $q = \phi_{-t}(p) \in A$ since A is positive invariant and $-t > 0$; this is a contradiction since $q \in A^c$. Thus $\phi_t(q) \in A^c$ $\forall t < 0$, i.e. A^c is negative invariant.

2.2 Chapter 4

Question 3. Consider the function $x(t) = e^{At}x_0$. This function (i) exists for all time, (ii) satisfies the ODE (including initial condition), (iii) is infinitely differentiable with respect to the initial condition (although all derivatives higher than the first vanish).

2.3 Chapter 5

Question 3. (a) We calculate

$$\begin{pmatrix}-1 & -1\\ 9 & -1\end{pmatrix}\begin{pmatrix}1\\ 3i\end{pmatrix} = \begin{pmatrix}-1-3i\\ 9-3i\end{pmatrix} = (-1-3i)\begin{pmatrix}1\\ 3i\end{pmatrix},$$

$$\begin{pmatrix}-1 & -1\\ 9 & -1\end{pmatrix}\begin{pmatrix}1\\ -3i\end{pmatrix} = \begin{pmatrix}-1+3i\\ 9+3i\end{pmatrix} = (-1+3i)\begin{pmatrix}1\\ -3i\end{pmatrix}.$$

(b) We calculate

$$
\begin{aligned}
T_1^{-1}AT_1 &= \begin{pmatrix}1 & 0\\ 0 & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}-1 & -1\\ 9 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -3\end{pmatrix} = \begin{pmatrix}-1 & 3\\ -3 & -1\end{pmatrix},\\
T_2^{-1}AT_2 &= \begin{pmatrix}1 & 0\\ 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}-1 & -1\\ 9 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix} = \begin{pmatrix}-1 & -3\\ 3 & -1\end{pmatrix},\\
T_3^{-1}AT_3 &= \begin{pmatrix}0 & -\frac{1}{3}\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1 & -1\\ 9 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ -3 & 0\end{pmatrix} = \begin{pmatrix}-1 & -3\\ 3 & -1\end{pmatrix},\\
T_4^{-1}AT_4 &= \begin{pmatrix}0 & \frac{1}{3}\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1 & -1\\ 9 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 3 & 0\end{pmatrix} = \begin{pmatrix}-1 & 3\\ -3 & -1\end{pmatrix}.
\end{aligned}
$$

(c) Let $u \pm iv$ be the eigenvectors of A. Then T above is one of $(u \mid -v)$, $(u \mid v)$, $(-v \mid u)$ and $(v \mid u)$.

2.4 Chapter 6

Question 1. (a) Let $p \in W^s(\bar{x})$. Then there exists $q \in W^s_{\text{loc}}(\bar{x})$ and $T < 0$ such that $p = \phi_T(q)$. Thus $\phi_t(p) = \phi_{t+T}(q) \in W^s(\bar{x})$. Thus $W^s(\bar{x})$ is invariant. A similar argument shows that $W^s(\bar{x})$ is invariant.

(b) Let $p \in W^s(\bar{x})$. Then there exists $q \in W^s_{\text{loc}}(\bar{x})$ and $T > 0$ such that $q = \phi_T(p)$. By definition of $W^s_{\text{loc}}(\bar{x})$, $\phi_t(q) \to \bar{x}$ as $t \to \infty$. Since $\phi_t(p) = \phi_{t-T}(q)$, it follows that $\phi_t(p) \to \bar{x}$ as $t \to \infty$. A similar argument holds for (c).

Question 2. The stable and unstable manifolds cannot intersect at an *isolated* point as illustrated in the figure since that would violate the uniqueness of the solution at the point.

However the global stable manifold can coincide with the global unstable manifold – this occurs for a homoclinic orbit.